Published by Institute of Physics Publishing for SISSA/ISAS
Received: *April 10, 2003*
Accepted: *July 13, 2003*

Fermionic ghosts in Moyal string field theory

Itzhak Bars
Department of Physics and Astronomy, University of Southern California
Los Angeles, CA 90089-0484, USA
E-mail: bars@usc.edu

Isao Kishimoto and Yutaka Matsuo
Department of Physics, Faculty of Science, University of Tokyo
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan
E-mail: ikishimo@hep-th.phys.s.u-tokyo.ac.jp, matsuo@phys.s.u-tokyo.ac.jp

Abstract: We complete the construction of the Moyal star formulation of bosonic open string field theory (MSFT) by providing a detailed study of the fermionic ghost sector. In particular, as in the case of the matter sector, (1) we construct a map from Witten's star product to the Moyal product, (2) we propose a regularization scheme which is consistent with the matter sector and (3) as a check of the formalism, we derive the ghost Neumann coefficients algebraically directly from the Moyal product. The latter satisfy the Gross-Jevicki nonlinear relations even in the presence of the regulator, and when the regulator is removed they coincide numerically with the expression derived from conformal field theory. After this basic construction, we derive a regularized action of string field theory in the Siegel gauge and define the Feynman rules. We give explicitly the analytic expression of the off-shell four point function for tachyons, including the ghost contribution. Some of the results in this paper have already been used in our previous publications. This paper provides the technical details of the computations which were omitted there.

Keywords: Bosonic Strings, String Field Theory, Non-Commutative Geometry.

JHEP07(2003)027

Contents

1. Introduction

Witten's open bosonic string field theory [1], its operator version [2], and its split string reformulations [3]–[7], led to the development of the Moyal star formulation of string field theory (MSFT) during the past two years ([7]–[12]). This formulation has the following features:

1. The star product in Witten's string field theory is mapped to the Moyal star product [7] after an appropriate change of variables. The string field $A(\bar{x}, x_e.p_e)$ in the new basis is a function of the string midpoint $\bar{x}$ and the phase space (x_e, p_e) of even string modes $e = 2, 4, 6, \ldots$. While Witten's star product, written in terms of Neumann coefficients [2], is very complicated to manipulate in computations, the Moyal star product in MSFT, which is diagonal in mode space labelled by e, is the simplest form of the star product that occurs in standard noncommutative geometry. This feature vastly simplifies the structure of string field theory and is very helpful in explicit computations.

2. The change of variables introduces a set of simple but infinite matrices T_{eo}, R_{oe}, v_o, w_e, labelled by $e = 2, 4, 6, \ldots$, and $o = 1, 3, 5, \ldots$, which contain basic information about even (e) and odd (o) string modes. These matrices obey a matrix algebra that has an associativity anomaly, which in turn feeds into an associativity anomaly among string fields [8]. The origin of the anomaly is the infinite number of string modes that cause the appearance of ambiguous terms of the form ∞/∞. In order to resolve this problem, we proposed [8, 9] a regularization by truncating the number of the oscillators to finite $2N$, and defined a deformed set of finite matrices T, R, v, w as functions of the oscillator frequencies κ_e, κ_o of the $2N$ modes. After such regulation, the associativity is restored and all manipulations become well-defined. The original open string field theory is restored by taking the original frequencies, $\kappa_e = e$, $\kappa_o = o$, and the large-N limit at the end of the computation. Through explicit computation of specific examples, it has been shown that this regulation procedure correctly reproduces results computed independently in conformal string theory.

3. In the regularized basis, we computed the Neumann coefficients analytically by using only the Moyal product. These coefficients are not needed for computations in MSFT, but they provide a check of MSFT relative to the operator formulation given in [2]. We have shown that the Neumann coefficients derived in the regularized MSFT framework satisfy the Gross-Jevicki nonlinear relations, and thus provide a generalization of Neumann coefficients for any set of frequencies κ_e, κ_o and any N. This provided the first consistency check of MSFT [9]. Furthermore, we found that the Neumann coefficients for any n-point vertex are all simple functions of a single matrix $t_{eo} = \kappa_e^{1/2} T_{eo} \kappa_o^{-1/2}$. Diagonalizing the matrix t diagonalizes all the Neumann coefficients simultaneously for all n-point vertices [9]. At large-N our diagonal form agrees with the one given in [13, 14], and explains in particular why there is Neumann spectroscopy for the 3-point vertex [13], and generalizes it to any n-point vertex [9].

4. One of the nice features of MSFT is that the star product is diagonal in mode space (i.e. independent and same for each mode). The only cost for this simplification is that the kinetic term given by the Virasoro operator L_0 becomes off-diagonal in mode space (x_e, p_e) [9]. However, this has not hindered explicit computations. In particular, we have derived the Feynman rules, including the propagator $(L_0 - 1)^{-1}$, and shown that we can evaluate efficiently and explicitly the Feynman graphs in open string field theory [10].

5. The off-diagonal part of the kinetic term depends on a specific combination of momentum modes, namely $\hat{p} = (1 + \bar{w}w)^{-1/2} \sum_e w_e p_e$ which we refer to as the "anomalous midpoint mode" [8, 11]. This mode appears in the kinetic operator in the form $L_0 = \gamma + \cdots$, with $\gamma \sim \hat{p}^2$, while the remaining part of the kinetic term is diagonal in mode space. We named the term γ the "midpoint correction". We found that if it were not for this midpoint correction, the rest of the kinetic plus interaction terms would define a theory, equivalent to an infinite matrix theory, that is vastly simpler and completely solvable [12]. However we determined that the γ term is essential for the correct definition of string field theory. Thus we have isolated the hard part of string field theory in the form of the quadratic "midpoint correction" term γ. We have shown that all classical solutions, including the true vacuum, of the interacting string theory, are obtained analytically by first solving the nonlinear equation explicitly by ignoring γ, and then including the effect of γ in a closed formal expression that can be evaluated to any order in a perturbative expansion in powers of γ [12].

In our published work so far, we have demonstrated all of the above results explicitly mainly in the matter sector. We have implied, and sometimes explicitly included, the corresponding contribution of ghosts either in the bosonic [9] or fermionic [10, 12] version but the details were not given explicitly. The purpose of the present paper is to provide all the relevant material on fermionic ghosts which we used previously, in an organized and comprehensive manner. In this sense, this paper completes the basic formulation of MSFT.

JHEP07(2003)027

Because we will try to be quite explicit, the content of this paper will be rather technical. The construction of the Moyal product for fermionic ghosts is basically parallel to the bosonic case [7, 8, 9], except that we need to be careful in some minor differences, including midpoint issues, which appear in the ghost fields b, c.

The first point relates to the boundary conditions. While we needed to consider Neumann type boundary conditions for the matter fields (open strings on the D25 brane), Dirichlet type boundary conditions appear for the ghost field. Therefore, the Fourier basis is different. The regularization method developed in [8, 9] was based on the Fourier modes, hence some care is needed to make the regularization compatible in the matter and ghost sectors. The regularization developed in this paper can be applied also to the treatment of the matter sector for lower Dp branes ($p < 25$).

The second point relates to the overlapping conditions of split strings. For the matter fields, we have to treat only overlapping conditions for the split string degrees of freedom. For the ghost fields, on the other hand, we should also consider the anti-overlapping conditions. This induces some changes in the mapping of Witten's star to Moyal's star.

The third point is the fermionic nature of Moyal variables. The usual bosonic derivatives which appear in the definition of the Moyal product get replaced by derivatives of fermionic variables, and care is needed in the ordering and signs.

The fourth point is the treatment of the midpoint mode. This is a rather delicate issue since, as in the matter sector, it cannot be determined from the split string formulation. The fermionic midpoint mode is not part of the Moyal $\star$ product, and it is integrated in the definition of the action. In the Siegel gauge, the dependence on this extra fermionic variable becomes trivial, and it drops out in actual computations.

We mention the work of Erler [16] where he defined the Moyal star formulation for the ghost system mainly in the continuous basis [14].[1] There are overlaps of the current paper with his work especially in the second and third points mentioned above. The correct treatment of the midpoint appeared first in our work [10], and his paper was modified subsequently. Beyond the preliminary level, for correct computation with ghosts, the results of the present paper are needed.

The definition of the fermionic Moyal product outlined above is given in section 2. Because of its technical nature, we first summarize the fundamental formulae at the beginning of the section, and explain the full detail in the subsections. Readers who wish to skip the details of the derivation can proceed to later sections by skipping the latter part. In subsection 2.1, we first discuss the issue of boundary conditions in general for fermions and the corresponding regularization scheme. We also give a brief review of [9] in appendix A. Together with it, this paper provides the basic formulae for both the ghost and matter sectors in a compact form. We then discuss the mapping from Witten's star to Moyal's star in subsection 2.2. Finally we apply these techniques to the actual bc ghost system in subsection 2.3, and derive the correspondence between the conventional oscillators and Moyal variables in subsection 2.4.

[1]There are some works on Moyal structure of Witten's string field theory. [15]

JHEP07(2003)027

After this preparation, in section 3, we define the monoid algebra among gaussian string fields, which provides a useful tool for computations (this is almost a group, except for inverse). We compute the star product of n monoid elements, which as a by-product give the Neumann coefficients in the ghost sector. These were conjectured in [9] by using a nontrivial relation with the Neumann coefficients in the matter sector. In this paper, we derive them directly from the fermionic Moyal product. They satisfy the Gross-Jevicki-nonlinear relations exactly for any frequencies κ_e, κ_o and any N. This fact confirms the consistency of our construction including the midpoint prescription. As in the matter sector, they are simple functions of the matrix t_{eo} for any n-point vertex, and they can be related to the corresponding matter Neumann coefficients by the simple procedure of replacing the matrix t by its inverse.

We also give the direct numerical comparison between the analytic form of the Neumann coefficients $\mathcal{M}$ obtained from conformal field theory and our algebraic expression. We confirmed that the approximate value for finite N converges to its exact value as $N \to \infty$ with the following universal behavior, $\mathcal{M}_{nm}(N)/\mathcal{M}_{nm}(cft) \sim 1 + a_{nm} N^{-\alpha}$ where the exponent is approximately $\alpha \sim 1.33$ for matter sector and $\alpha \sim 0.67$ for the ghost sector for any components of the Neumann coefficients. While this analysis is of different nature from the other parts in this paper, it is included here since it gives strong support on the consistency of MSFT in [9] and this paper. It also provides the basis for numerical computation of MSFT in our future study.

In section 4 we apply the formalism. First, we present the derivation of the open string field action in the Siegel gauge by including both matter and ghost fields. This action was the starting point in our recent work [10, 12] where we used the results of the present paper without providing the details. Finally, we also compute the ghost contribution to the Feynman graphs for the four-point scattering amplitude for off-shell tachyons, whose matter sector was discussed in [10].

2. Moyal's star from Witten's star in fermionic ghost sector

In this section we construct the map from Witten's star product to the Moyal star product for fermionic variables. The fermionic version of the Moyal $\star$ product was called "anti-Moyal star product" in the literature [18]. More general star products have also been considered in the context of deformation quantization of super-Poisson brackets [19]. The anti-Moyal star product will simply be referred to as the Moyal star product in the following. A first basic construction for the ghost bc system, following the one in the matter sector [7], was previously discussed in [16], while the correct treatment of the midpoint was first given in [10]. In this section we give the complete treatment, including the consistent regularization with the matter sector.

We also define the even basis of ghost modes that is most transparent for our computations, after defining some other bases as well. We first summarize the main results of this rather lengthy and technical section:

- The Moyal $\star$ acts on fermionic ghost modes $\xi \equiv (x_o, p_o, y_o, q_o)$ $(o = 1, 3, 5, \ldots, 2N - 1)$ as

$$(A \star B)(x_o, p_o, y_o, q_o) = A \exp\left(\frac{\theta'}{2}\sum_{o>0}\left(\frac{\overleftarrow{\partial}}{\partial x_o}\frac{\overrightarrow{\partial}}{\partial p_o}+\frac{\overleftarrow{\partial}}{\partial y_o}\frac{\overrightarrow{\partial}}{\partial q_o}+\frac{\overleftarrow{\partial}}{\partial p_o}\frac{\overrightarrow{\partial}}{\partial x_o}+\frac{\overleftarrow{\partial}}{\partial q_o}\frac{\overrightarrow{\partial}}{\partial y_o}\right)\right) B\,, \tag{2.1}$$

where θ' is a parameter which absorbs units, and if desired, could be absorbed away by a rescaling of the variables which amounts to a choice of units. We note the canonical structure $\{x_o, p_{o'}\}_\star = \theta' \delta_{oo'} = \{y_o, q_{o'}\}$. This odd basis (o) of ghost modes, which was used in [10], is naturally defined in the process of mapping the Witten star to the Moyal star, including the treatment of the midpoint. However, a more transparent basis that is more parallel to the matter sector, which simplifies the overall formalism, is obtained by rewriting the odd basis, through the following linear *canonical* transformation, in terms of an even basis $x_e^b, p_e^b, x_e^c, p_e^c$ $(e = 2, 4, \ldots, 2N)$, where the labels b, c refer to the modes of the usual b, c ghosts

$$x_e^b := \kappa_e^{-1}\sum_{o>0} S_{eo} x_o\,, \quad p_e^b := \kappa_e \sum_{o>0} S_{eo} p_o\,, \quad x_e^c := \sum_{o>0} T_{eo} y_o\,, \quad p_e^c := \sum_{o>0} q_o R_{oe}\,. \tag{2.2}$$

This even basis is different than the one defined in [16]. The Moyal product is rewritten as

$$(A \star B)(x_e^b, p_e^b, x_e^c, p_e^c) = A \exp\left(\frac{\theta'}{2}\sum_{e>0}\left(\frac{\overleftarrow{\partial}}{\partial x_e^b}\frac{\overrightarrow{\partial}}{\partial p_e^b}+\frac{\overleftarrow{\partial}}{\partial x_e^c}\frac{\overrightarrow{\partial}}{\partial p_e^c}+\frac{\overleftarrow{\partial}}{\partial p_e^b}\frac{\overrightarrow{\partial}}{\partial x_e^b}+\frac{\overleftarrow{\partial}}{\partial p_e^c}\frac{\overrightarrow{\partial}}{\partial x_e^c}\right)\right) B\,. \tag{2.3}$$

We note the canonical structure $\left\{x_e^b, p_{e'}^b\right\}_\star = \theta' \delta_{ee'} = \left\{x_e^c, p_{e'}^c\right\}_\star$. The linear transformation matrices, T, R (and matrices U and vectors v, w which appear in the following) were defined in [9], while the matrix S appears for the first time in this paper. Their properties are derived explicitly in subsection 2.1. They play a central role in MSFT since they define a Bogoliubov transformation from the oscillators in the operator formalism to the Moyal coordinates, and thus carry essential information about string theory. For instance, see eqs. (2.4), (2.6) in the next paragraph. For the moment, we just mention that they satisfy the inverse properties $TR = RT = S\bar{S} = \bar{S}S = 1$ which prove that the two versions of the star product (2.1) and (2.3) are canonically equivalent. Throughout this paper, the bar $(\bar{\ })$ means the transpose of matrices or vectors. The relation of split strings to the Moyal star is discussed in subsection 2.2.

- In the operator formulation, the ghost sector of the string field $|\Psi\rangle$ is represented in the Fock space of the bc ghost oscillators. The transformation to the Moyal field $A(\xi)$ as a function of noncommutative coordinates $\xi = (x, p)$ is obtained through the Fourier transformation [7] of the coordinate representation of $|\Psi\rangle$. The whole procedure is more neatly expressed, as in the matter sector [9], by the inner product with a particular bra state $\langle \xi_0, \xi_1, \xi_2|$ in the Fock space, where $\bar{\xi}_1 = (x_o, p_o)\,, \bar{\xi}_2 = (y_o, q_o)$ are the noncommutative fermionic coordinates, and ξ_0 is a fermionic variable related to the zero mode dependence of $|\Psi\rangle$

$$\begin{aligned}\langle\xi_0,\xi_1,\xi_2| &= -2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}\times \\ &\quad\times\langle\Omega|\hat{c}_{-1}e^{-\xi_0(\hat{c}_0-\sqrt{2}\bar{w}\hat{c}_e)}e^{\hat{c}_e\hat{b}_e-\hat{c}_o\hat{b}_o-2i\bar{\xi}1M_0^{(o)}\xi_2-\bar{\xi}_1\lambda_1-\bar{\xi}_2\lambda_2}\,, \qquad (2.4)\\ \hat{A}(\xi_0,\xi_1,\xi_2) &= \langle\xi_0,\xi_1,\xi_2|\Psi\rangle\,. \qquad (2.5)\end{aligned}$$

Here $\langle\Omega|$ is the $\mathrm{SL}(2,\mathrm{R})$ invariant bra Fock vacuum, $\hat{b}_n,\hat{c}_n$ are the conventional ghost oscillators, and the matrices $M_0^{(o)}$ and λ are defined as

$$M_0^{(o)}=\begin{pmatrix}\frac{1}{2}\mathbf{1}_o & 0\\ 0 & \frac{2}{\theta'^2}\left(\bar{S}\bar{R}\right)_{oo'}\end{pmatrix},\quad \lambda_1=\begin{pmatrix}-i\sqrt{2}\hat{c}_o\\ \frac{2i}{\theta'}\bar{S}_{oe}\left(-\sqrt{2}\hat{b}_e+w_e\xi_0\right)\end{pmatrix},\quad \lambda_2=\begin{pmatrix}-\sqrt{2}\hat{b}_o\\ -\frac{2\sqrt{2}}{\theta'}R_{oe}\hat{c}_e\end{pmatrix}. \qquad (2.6)$$

The matrices w_e,S_{eo},R_{oe} are functions of the oscillator frequencies κ_e,κ_o as given below. Through eqs. (2.1, 2.4) we map Witten's star ($\star^W$) into the Moyal's star $\star$ as follows

$$\begin{aligned}\langle\xi_0,\xi|\Psi_1\star^W\Psi_2\rangle &\sim \langle\xi_0,\xi|\Psi_1\rangle\star\langle\xi_0,\xi|\Psi_2\rangle\,, \qquad (2.7)\\ |\Psi_1\star^W\Psi_2\rangle_3 &= {}_1\langle\Psi_1|_2\langle\Psi_2|V_3\rangle_{123}\,,\quad {}_1\langle\Psi_1|={}_{14}\langle V_2|\Psi_1\rangle_4,\ {}_2\langle\Psi_2|={}_{25}\langle V_2|\Psi_2\rangle_5\,. \qquad (2.8)\end{aligned}$$

We note that the product is local in ξ_0, while ξ_0 plays a similar role to the midpoint coordinate $\bar{x}^\mu$ in the matter sector. The Moyal star reproduces correctly the three string vertex $|V_3\rangle$ and the reflector $|V_2\rangle$ of the operator formalism [2]. The details are given in subsection 2.3. The precise correspondence including the zero mode is in section 3.

- Eq. (2.4) is enough to derive the connection between the conventional operator formalism and the Moyal star formalism. For example, the action of the standard oscillators on the Fock space field $|\Psi\rangle$ can be rewritten in terms of their Moyal images acting on the field $\hat{A}=\langle\xi_0,\xi_1,\xi_2|\Psi\rangle$ through the star product, as follows

$$\begin{aligned}\hat{b}_0|\Psi\rangle &\leftrightarrow -\xi_0\hat{A}\,,\\ \hat{c}_0|\Psi\rangle &\leftrightarrow \left(-\frac{\partial}{\partial\xi_0}+\frac{\theta'}{2}\bar{v}\frac{\partial}{\partial q_o}\right)\hat{A}\,,\\ \hat{b}_o|\Psi\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\left(\beta_o^b\star\hat{A}-(-1)^{|A|}\hat{A}\star\beta_{-o}^b\right),\\ \hat{c}_o|\Psi\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\left(\beta_o^c\star\hat{A}+(-1)^{|A|}\hat{A}\star\beta_{-o}^c\right),\\ \hat{b}_e|\Psi\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\left(\beta_e^b\star\hat{A}+(-1)^{|A|}\hat{A}\star\beta_{-e}^b\right)+w'_e\,\xi_0\hat{A}\,,\\ \hat{c}_e|\Psi\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\left(\beta_e^c\star\hat{A}-(-1)^{|A|}\hat{A}\star\beta_{-e}^c\right), \qquad (2.9)\end{aligned}$$

where β_o^b,β_o^c are fields in Moyal space which obey oscillator relations under the $\star$ product

$$\beta_o^b:=\frac{1}{2}\left(\frac{2}{\theta'}q_{|o|}-i\epsilon(o)x_{|o|}\right),\qquad \beta_o^c:=\frac{1}{2}\left(y_{|o|}-i\frac{2}{\theta'}\epsilon(o)p_{|o|}\right),\quad \{\beta_o^b,\beta_{o'}^c\}_\star=\delta_{o+o'}\,. \qquad (2.10)$$

JHEP07(2003)027

On the other hand β_e^b, β_e^c are not independent from the β_o^b, β_o^c. Rather, they are their Bogoliubov transforms which obey the following relations

$$\beta_e^b = \sum_o \beta_o^b U^{-1}_{-o,e}\,, \qquad \beta_e^c = \sum_o U_{e,-o}\beta_o^c\,,\ \{\beta_e^b, \beta_{e'}^c\}_\star = \delta_{e+e'}\,, \tag{2.11}$$

and $\{\beta_{-o}^b, \beta_e^c\}_\star = U_{e,-o}$ where the matrix $U_{e,-o}$ will be given below. With these formulas, one can directly translate operators in Fock space into their images which act in Moyal space. The proof of these formulas and some variants are discussed in subsection 2.4. In this way we derive the explicit form of the Virasoro operator L_0 which acts in Moyal space

$$\begin{aligned} L_0 &= \sum_{k=1}^{2N} \kappa_k (\hat\beta^b_{-k}\hat\beta^c_k + \hat\beta^c_{-k}\hat\beta^b_k) \qquad\qquad (2.12)\\ &= \sum_{k=1}^{2N}\kappa_k + i\sum_{o>0}\kappa_o\left(x_o y_o + \frac{\partial}{\partial x_o}\frac{\partial}{\partial y_o}\right) + i\sum_{o>0}\kappa_o\left(\frac{4}{\theta'^2}p_o q_o + \frac{\theta'^2}{4}\frac{\partial}{\partial p_o}\frac{\partial}{\partial q_o}\right) + \\ &\quad + \frac{4i}{\theta'^2}(1+\bar w w)\left(\sum_{o>0}\kappa_o v_o p_o\right)\left(\sum_{o'>0} v_{o'} q_{o'}\right) + \frac{2i}{\theta'}(1+\bar w w)\left(\sum_{o>0} v_o\kappa_o p_o\right)\xi_0\,. \end{aligned}$$

This was used in [10, 12]. Here $\hat\beta_k^b, \hat\beta_k^c$ are not the Moyal fields β_o^b, β_o^c or β_e^b, β_e^c given in eqs. (2.10), (2.11); rather, they are differential operators that obey the standard oscillator relations, and which are derived from the star products of the fields β_o^b, β_o^c or β_e^b, β_e^c as will be shown below.

2.1 Half string formalism and regularization

We start from full string functions $\psi(\sigma)$, $0 \le \sigma \le \pi$ with Dirichlet boundary conditions at $\sigma = 0, \pi$, and discuss their split string formulation in the interval $0 \le \sigma \le \pi/2$. Such functions have a Fourier expansion with only *sine* modes in the full string formalism. By contrast, the corresponding problem in the matter sector involved only *cosine* modes because of Neumann boundary conditions at $\sigma = 0, \pi$. We collect the basic formulae in the appendix A.1. The essential step was the construction of the regularization [8, 9] which is needed to avoid the associativity anomaly. We give a regularization of the ghost sector for the sine mode expansion which is compatible with the previous results.

A full string function $\psi(\sigma)$ which satisfies Dirichlet boundary conditions $\psi(0) = \psi(\pi) = 0$ is expanded as

$$\psi(\sigma) = \sqrt{2}\sum_{n=1}^{\infty}\psi_n \sin n\sigma\,,\ \psi_n = \frac{\sqrt{2}}{\pi}\int_0^\pi d\sigma\psi(\sigma)\sin n\sigma\,. \tag{2.13}$$

We decompose such a field $\psi(\sigma)$ into left half $l(\sigma)$ and right half $r(\sigma)$ as follows

$$\psi(\sigma) = \begin{cases} l(\sigma) & 0 \le \sigma \le \frac{\pi}{2} \\ r(\pi-\sigma) & \frac{\pi}{2} \le \sigma \le \pi \end{cases}\,,\ \psi_n = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\sigma(l(\sigma) - (-1)^n r(\sigma))\sin n\sigma\,. \tag{2.14}$$

As we have seen in [7, 8], we have some arbitrariness in the choice of the boundary condition for the split string functions $l(\sigma), r(\sigma)$ at the midpoint $\sigma = \pi/2$. They can be expanded using either odd or even modes according to the two possible choices of the boundary condition at the midpoint $\sigma = \pi/2$ as discussed below.

2.1.1 Dirichlet at end point, Neumann at midpoint (DN)

First we consider Neumann boundary conditions at the midpoint $\sigma = \pi/2$, while we have Dirichlet boundary conditions at the end point for the split string functions $l(\sigma), r(\sigma)$:

$$l(0) = r(0) = 0\,, \qquad l'\left(\frac{\pi}{2}\right) = r'\left(\frac{\pi}{2}\right) = 0\,. \tag{2.15}$$

We can expand them using odd sine modes $o = 1, 3, 5, \ldots$

$$l(\sigma) = \sqrt{2}\sum_{o=1}^{\infty} l_o \sin o\sigma\,, \qquad l_o = \frac{2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\sigma\, l(\sigma)\sin o\sigma\,, \tag{2.16}$$

$$r(\sigma) = \sqrt{2}\sum_{o=1}^{\infty} r_o \sin o\sigma\,, \qquad r_o = \frac{2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\sigma\, r(\sigma)\sin o\sigma\,. \tag{2.17}$$

By comparing the mode expansions eqs. (2.13), (2.16), (2.17) with (2.14), we obtain the correspondence between the full and split string variables,

$$l_o = \psi_o + \bar{S}_{oe}\psi_e\,, \qquad r_o = \psi_o - \bar{S}_{oe}\psi_e\,, \tag{2.18}$$

or the inverse

$$\psi_e = \frac{1}{2}S_{eo}(l_o - r_o)\,, \qquad \psi_o = \frac{1}{2}(l_o + r_o)\,, \tag{2.19}$$

where $e = 2, 4, 6, \ldots$, and the matrix S_{eo} is given by

$$S_{eo} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} d\sigma \sin e\sigma \sin o\sigma = \frac{4 i^{o-e+1} e}{\pi(e^2 - o^2)}\,. \tag{2.20}$$

The above mappings are consistent because S_{eo} is an orthogonal matrix:

$$\bar{S}S = S\bar{S} = 1\,. \tag{2.21}$$

The continuity condition at the midpoint $\psi(\pi/2) = l(\pi/2) = r(\pi/2)$ is

$$\sum_{o=1}^{\infty} \tilde{w}_o\psi_o = \sum_{o=1}^{\infty} \tilde{w}_o l_o = \sum_{n=1}^{\infty} \tilde{w}_o r_o \tag{2.22}$$

where we defined the odd vector $\tilde{w}$ associated with the midpoint

$$\tilde{w}_o = \sqrt{2}\sin\frac{o\pi}{2} = \sqrt{2}\, i^{o-1}\,. \tag{2.23}$$

Eq. (2.22) holds thanks to the identity $(S\tilde{w})_e = \sum_{o=1}^{\infty} S_{eo}\tilde{w}_o = 0$. This equation implies that S has a singular eigenvector even though it has an inverse, which is just its transpose $\bar{S}$, as stated in (2.21). This esoteric relation is possible because S is an infinite matrix.

However it causes an associativity anomaly with respect to matrix products, which in turn feeds into associativity anomaly of string field star products, as discussed in [8].

An example of the associativity anomaly is $\bar{S}(S\tilde{w}) = 0$, but $(\bar{S}S)\tilde{w} = \tilde{w}$. Each single sum indicated by the parentheses has a unique answer, but the double sums are ambiguous. The reason is that, due to infinite sums, in the first expression there are terms of the form ∞/∞ which are ambiguous. Since these matrices appear in many physical computations we must give an unambiguous definition of the matrix product. We will resolve this ambiguity in computations successfully by a regularization procedure.

2.1.2 Dirichlet at end point, Dirichlet at midpoint (DD)

We consider another possibility: we define the midpoint of the string $\bar{\psi} := \psi(\pi/2)$, and impose Dirichlet boundary conditions at both $\sigma = 0, \pi/2$ on the split string functions $l(\sigma), r(\sigma)$,

$$l(0) = r(0) = 0\,, \qquad l\left(\frac{\pi}{2}\right) = r\left(\frac{\pi}{2}\right) = \bar{\psi}\,. \tag{2.24}$$

We note that the midpoint value $\bar{\psi}$ is an additional degree of freedom in the split string basis and cannot be chosen arbitrarily. We expand $l(\sigma), r(\sigma)$ using even sine modes, $e = 2, 4, 6, \ldots$

$$l(\sigma) = \frac{2}{\pi}\sigma\bar{\psi} + \sqrt{2}\sum_{e=2}^{\infty} l_e \sin e\sigma\,, \qquad l_e = \frac{2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\sigma\left(l(\sigma) - \frac{2}{\pi}\sigma\bar{\psi}\right)\sin e\sigma\,, \tag{2.25}$$

$$r(\sigma) = \frac{2}{\pi}\sigma\bar{\psi} + \sqrt{2}\sum_{e=2}^{\infty} r_e \sin e\sigma\,, \qquad r_e = \frac{2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\sigma\left(r(\sigma) - \frac{2}{\pi}\sigma\bar{\psi}\right)\sin e\sigma\,. \tag{2.26}$$

Again, by comparing the mode expansions eqs. (2.13), (2.25, 2.26) with (2.14), the correspondence between split and full string variables is obtained

$$\bar{\psi} = \tilde{w}_o\psi_o\,, \qquad l_e = \psi_e + \tilde{T}_{eo}\psi_o\,, \qquad r_e = -\psi_e + \tilde{T}_{eo}\psi_o\,, \tag{2.27}$$

or the inverse

$$\psi_e = \frac{1}{2}(l_e - r_e)\,, \qquad \psi_o = \tilde{u}_o\bar{\psi} + \frac{1}{2}\bar{S}_{oe}(l_e + r_e)\,, \tag{2.28}$$

where

$$\tilde{T}_{eo} = \frac{4o^2\ i^{o-e+1}}{\pi e(e^2 - o^2)} = S_{eo} + \tilde{v}_e\tilde{w}_o = \frac{1}{e^2}S_{eo}o^2\,, \tag{2.29}$$

$$\tilde{u}_o = \frac{4\sqrt{2}}{\pi^2}\int_0^{\frac{\pi}{2}} d\sigma\sigma\sin o\sigma = \frac{4\sqrt{2}\, i^{o-1}}{\pi^2 o^2}\,, \tag{2.30}$$

$$\tilde{v}_e = -\frac{4\sqrt{2}}{\pi^2}\int_0^{\frac{\pi}{2}} d\sigma\sigma\sin e\sigma = \frac{2\sqrt{2}\, i^e}{e\pi}\,. \tag{2.31}$$

The maps $(\bar{\psi}, l_e, r_e) \leftrightarrow (\psi_n)$ are consistent by the relation

$$\tilde{u}_o\tilde{w}_o = 1\,, \qquad \tilde{T}\bar{S} = 1\,, \qquad \tilde{T}\tilde{u} = 0\,, \qquad S\tilde{w} = 0\,. \tag{2.32}$$

We can prove the following relations among infinite matrices $S, \tilde{T}$ and vectors $\tilde{u}, \tilde{v}, \tilde{w}$ by straightforward computation:

$$S = \kappa_e^2\,\tilde{T}\,\kappa_o^{-2}\,, \quad S = \tilde{T} - \tilde{v}\bar{\tilde{w}}\,, \quad \bar{S}\tilde{v} = -\tilde{u}\,, \quad S\tilde{u} = -\tilde{v}\,, \quad \bar{\tilde{T}}\tilde{v} = -\tilde{u} + \frac{1}{3}\tilde{w}\,, \tag{2.33}$$

$$\bar{S}S = S\bar{S} = \tilde{T}\bar{S} = 1\,, \quad \bar{S}\tilde{T} = 1 - \tilde{u}\bar{\tilde{w}}\,, \quad S\tilde{w} = \tilde{T}\tilde{u} = 0\,, \quad \bar{\tilde{w}}\tilde{u} = 1\,, \quad \bar{\tilde{v}}\tilde{v} = \bar{\tilde{u}}\tilde{u} = \frac{1}{3}\,, \tag{2.34}$$

where κ_e, κ_o are the diagonal matrices $\kappa_e = \mathrm{diag}(2, 4, 6\ldots)$ and $\kappa_o = \mathrm{diag}(1, 3, 5, \ldots)$. This algebra is similar to the one among the infinite matrices T, R, v, w which appeared in the matter sector [8], where a full string function $\psi(\sigma)$ is expanded in terms of cosine modes (see appendix A.1).

2.1.3 Regularization

In the split string formulation given in appendix A.1.1 appendix A.1.2 section 2.1.1 section 2.1.2, we encountered a set of infinite dimensional matrices $T, R, S, \tilde{T}$ and vectors $w, v, \tilde{w}, \tilde{v}, \tilde{u}$. These represent Bogoliubov transformations between odd and even modes, with (T, R, w, v) appearing when the full string is expanded in terms of cosine modes, and $(S, \tilde{T}, \tilde{w}, \tilde{v}, \tilde{u})$ appearing when the full string is expanded in terms of sine modes. Such transformations are essential in the Moyal formulation since they carry basic information about string theory. We note that, the Moyal star product itself, which is applied independently for each mode, has no specific information about string theory, and as such is a more general structure.

In the analysis of the matter sector [8] as well as the sine mode expansion given so far, there appears an associativity anomaly in the matrix algebra of these matrices. This originates from the infinite dimensionality of these matrices. It produces ambiguities in computations in string field theory. To have a well defined theory, it is mandatory to define a deformed, unambiguous, associative algebra that preserves the basic matrix algebraic structure of these matrices [8, 9]. Such a deformation contains a parameter N, that corresponds to the rank of the matrices. The original definition of these matrices given above is reproduced by taking the limit $N \to \infty$ of this parameter. All computations in the open string field theory are performed unambiguously with finite N, and the correct value in string theory is obtained at the end of the computation by taking the limit $N \to \infty$. This is the basic strategy for practical computations in the MSFT proposal.

The deformed set of matrices for T, R, w, v that have correctly reproduced string theory was proposed in [8, 9]. Since an additional set of matrices $S, \tilde{T}, \tilde{w}, \tilde{v}, \tilde{u}$ have appeared in the ghost sector we need to obtain their deformation consistently with the matter sector. For that purpose, we start from the infinite dimensional matrix U and vectors w', v' defined in [9],

$$U_{-e,o} = \frac{2}{\pi}\frac{i^{o-e-1}}{o-e}\,, \qquad U^{-1}_{-o,e} = \frac{2}{\pi}\frac{e}{o}\frac{i^{o-e-1}}{o-e}\,, \qquad w'_e = i^{-e+2}\,, \qquad v'_o = \frac{2}{\pi}\frac{i^{o-1}}{o}\,, \tag{2.35}$$

where e (o) now run over both positive and negative integers $\pm2, \pm4, \ldots$ (resp. $\pm1, \pm3, \ldots$). In single sums these matrices satisfy

$$\sum_o U_{-e,o}U^{-1}_{-o,e'} = \delta_{e,e'}\,, \qquad \sum_{e\neq0} U^{-1}_{-o,e}U_{-e,o'} = \delta_{o,o'}\,, \tag{2.36}$$

which we denote $UU^{-1} = 1_e$, $U^{-1}U = 1_o$ for short in the following. More importantly, there exists the following matrix relations among them,

$$U^{-1} = {\kappa'_o}^{-1}\bar{U}\kappa'_e\,, \qquad U^{-1} = \bar{U} + v'\bar{w}'\,, \qquad v' = \bar{U}w'\,, \qquad w' = \bar{U}^{-1}v'\,, \tag{2.37}$$

where $\kappa'_e = \mathrm{diag}(\dots, -4, -2, 2, 4, \dots)$, and $\kappa'_o = \mathrm{diag}(\dots, -3, -1, 1, 3, \dots)$ are the diagonal matrices which specify the spectrum. These relations will be used as the defining relations. From them it is possible to derive the matrices themselves, as given in eq. (2.35). Therefore, they will be used as the basic relations that are also satisfied by the deformed matrices, as given below. The first relation implies that U defines an invertible Bogoliubov transformation between even and odd spectra (see also eq. (2.11)). The second relation shows that the transformation U is almost orthogonal except for the vectors v', w' which are associated with the midpoint mode. Finally the last two define the relation between the vectors.

On the other hand, the matrices (2.35) also satisfy

$$U\bar{U} = 1_e\,, \qquad Uv' = 0\,, \qquad \bar{v}'v' = 1\,, \tag{2.38}$$

which break the associativity [8, 9]. Therefore, these relations will be deformed in the regulated theory, as seen below. Of course these equations will hold when the regulator is removed.

All the other matrices are written in terms of U, w', v', (for $e, o > 0$),

$$T_{eo} = U_{-e,o} + U_{e,o}\,, \qquad R_{oe} = U^{-1}_{-o,e} + U^{-1}_{o,e}\,, \qquad w_e = \sqrt{2}w'_e\,, \qquad v_o = \sqrt{2}v'_o\,, \tag{2.39}$$

$$S_{eo} = U_{-e,o} - U_{e,o} = U^{-1}_{-o,e} - U^{-1}_{o,e}\,, \qquad \tilde{T}_{eo} = \kappa_e^{-1} T \kappa_o\,, \tag{2.40}$$

$$\tilde{u} = \frac{2}{\pi}\kappa_o^{-1} v\,, \qquad \tilde{w} = \frac{\pi}{2}\kappa_o v\,, \qquad \tilde{v} = -\frac{2}{\pi}\kappa_e^{-1} w\,, \tag{2.41}$$

where κ_o and κ_e are restrictions of κ'_o and κ'_e to the positive sector. These definitions in terms of U, w', v' together with the relations (2.37) among U, w', v' are sufficient to derive all the relations among $T, R, w, v, S, \tilde{T}, \tilde{u}, \tilde{v}, \tilde{w}$. Therefore, we may use these relations as the *definitions* of these matrices and vectors even when we use the regularization of U, w', v'.

In [8, 9, 10], the regularization of U, v', w' is given explicitly. We truncate the size of U, v', w' to $2N$ while keeping their property of Bogoliubov transformation between even and odd spectrum. It turns out that one may take the even and odd frequencies κ_e, κ_o as arbitrary functions of the positive integers (e, o), while keeping the reflection property of $\kappa'_{e,o}$ to extend the definition to negative integers $\kappa'_{-e} = -\kappa'_e$, $\kappa'_{-o} = -\kappa'_o$. Therefore we put

$$\kappa'_e = \epsilon(e)\kappa_{|e|}\,, \qquad \kappa'_o = \epsilon(o)\kappa_{|o|}\,. \tag{2.42}$$

We suppose implicitly that $\kappa'_e \neq 0, \kappa'_o \neq 0$ and that these are not degenerate.

The matrices U, U^{-1} and vectors w', v' can then be *derived* from the defining relations (2.37) as functions of the arbitrary spectral parameters κ_e, κ_o as (see appendix B for details of the derivation)

$$U_{-e,o} = \frac{w'_e v'_o \kappa'_o}{\kappa'_e - \kappa'_o}\,, \qquad U^{-1}_{-o,e} = \frac{w'_e v'_o \kappa'_e}{\kappa'_e - \kappa'_o}\,, U_{-e,o} = U_{e,-o}\,, \qquad U^{-1}_{-o,e} = U^{-1}_{o,-e}\,, \tag{2.43}$$

$$\sqrt{2}\,w'_e = w_{|e|} = i^{2-e}\frac{\prod_{o'>0}\left|\kappa^2_{|e|}/\kappa^2_{o'} - 1\right|^{\frac{1}{2}}}{\prod_{e'>0, e'\neq|e|}\left|\kappa^2_{|e|}/\kappa^2_{e'} - 1\right|^{\frac{1}{2}}}\,, \qquad w'_e = w'_{-e}\,, \tag{2.44}$$

$$\sqrt{2}\,v'_o = v_{|o|} = i^{|o|-1}\frac{\prod_{e'>0}\left|1-\kappa_{|o|}^2/\kappa_{e'}^2\right|^{\frac{1}{2}}}{\prod_{o'>0,o'\neq|o|}\left|1-\kappa_{|o|}^2/\kappa_{o'}^2\right|^{\frac{1}{2}}}\,,\qquad v'_o = v'_{-o} \tag{2.45}$$

where now the indices (e,o) run over the finite set $e = \pm2,\pm4,\ldots,\pm2N$ and $o = \pm1$, $\pm3,\ldots,\pm(2N-1)$. It is easy to check explicitly that in the limit $N\to\infty$ and $\kappa_e = e$, $\kappa_o = o$, these expressions reduce to eq. (2.35).

The regulated expressions for these matrices look considerably more complicated than their large-N limit. Therefore, it may appear that this would create a problem in analytic computations. Actually this is not the case at all, because in analytic computations one uses the matrix relations satisfied by these matrices rather than the explicit matrices themselves. The relations are preserved in the regularized version for any N, and they look the same as their $N=\infty$ counterpart. Therefore analytically the expressions in any computation look the same in the regulated or infinite versions as long as they are written in terms of these matrices without using their explicit form. The explicit construction of the regulated version insures associativity and eliminates the ambiguity of the associativity anomaly as explained below. Thus, in addition to the basic defining relations (2.37) which are the same for any N, including $N=\infty$, there are more relations among U,U^{-1},v',w' that can now be derived from the defining relations alone for any N,κ'_e,κ'_o:

$$UU^{-1} = 1\,,\qquad U^{-1}U = 1\,,\qquad \bar{U}^{-1}U^{-1} = 1+w'\bar{w}'\,,\qquad \bar{U}U = 1-v'\bar{v}'\,, \tag{2.46}$$

$$U\bar{U} = 1-\frac{w'\bar{w}'}{1+\bar{w}'w'}\,,\qquad Uv' = \frac{w'}{1+\bar{w}'w'}\,,\qquad \bar{v}'v' = \frac{\bar{w}'w'}{1+\bar{w}'w'}\,, \tag{2.47}$$

$$U^{-1}w' = v'(1+\bar{w}'w')\,,\qquad U^{-1}\bar{U}^{-1} = 1+v'\bar{v}'(1+\bar{w}'w')\,, \tag{2.48}$$

$$1+\bar{w}'w' = \frac{\prod_e \kappa'_e}{\prod_o \kappa'_o} = \frac{\prod_{e>0}\kappa_e^2}{\prod_{o>0}\kappa_o^2}\,. \tag{2.49}$$

In particular note that eqs. (2.38) are now deformed into eqs.(2.47), and that $\bar{w}'w'\to 2N\to\infty$ in the large-N limit. Thus, the deformed algebra actually holds also at $N=\infty$; it simply makes explicit the behavior as a function of N. With this, the associativity anomaly hidden in the original algebra is now resolved and the matrix algebra for all the matrices $U,U^{-1},T,R,S,\tilde{T},v,w,\tilde{v},\tilde{w},\tilde{u}$ defined through eqs. (2.39)–(2.45) becomes associative.

In particular, from eq. (2.39)–(2.41) we obtain the regularized matrices, such as T,R,S,

$$T_{eo} = \frac{w_e v_o \kappa_o^2}{\kappa_e^2-\kappa_o^2}\,,\qquad R_{oe} = \frac{w_e v_o \kappa_e^2}{\kappa_e^2-\kappa_o^2}\,,\qquad S_{eo} = \frac{w_e v_o \kappa_e\kappa_o}{\kappa_e^2-\kappa_o^2}\,. \tag{2.50}$$

From the relations among U,w',v' we can derive the relations among $T,R,v,w,S,\tilde{T},\tilde{u},\tilde{v},\tilde{w}$. The deformed algebra among T,R,w,v (A.18) is already given in [8], while the deformed algebra among $S,\tilde{T},\tilde{u},\tilde{v},\tilde{w}$ which replaces (2.33), (2.34) is,[2]

$$S = \kappa_e^2\tilde{T}\kappa_o^{-2}\,,\qquad S = \tilde{T}-\tilde{v}\bar{\tilde{w}}\,,\qquad \bar{S}\tilde{v} = -\tilde{u}\,,\qquad S\tilde{u} = -\tilde{v}\,, \tag{2.51}$$

[2]Note that for finite N, the continuity condition at the midpoint (2.22) is not satisfied because $S\tilde{w}\neq0$. However, we recover it, as well as all other infinite matrix relations by taking the open string limit $\kappa_e = e,\kappa_o = o,N=\infty$.

$$\bar{S} S = 1\,, \qquad S \bar{S} = 1\,, \qquad \bar{S}\tilde{T} = 1 - \tilde{u}\bar{\tilde{w}}\,, \qquad \tilde{T}\bar{S} = 1 - \frac{\kappa_e^{-1} w \bar{w} \kappa_e}{1 + \bar{w} w}\,, \tag{2.52}$$

$$S\tilde{w} = \frac{2}{\pi}\frac{\kappa_e w}{1 + \bar{w} w}\,, \qquad \tilde{T}\tilde{u} = \frac{2}{\pi}\frac{\kappa_e^{-1} w}{1 + \bar{w} w}\,, \qquad \bar{\tilde{T}}\tilde{v} = -\tilde{u} + \left(\frac{2}{\pi}\right)^2 \bar{w}\kappa_e^{-2} w \tilde{w}\,, \tag{2.53}$$

$$\bar{\tilde{w}}\tilde{u} = \bar{v} v = \frac{\bar{w} w}{1 + \bar{w} w}\,, \qquad \bar{\tilde{v}}\tilde{v} = \left(\frac{2}{\pi}\right)^2 \bar{w}\kappa_e^{-2} w\,, \qquad \bar{\tilde{u}}\tilde{u} = \left(\frac{2}{\pi}\right)^2 \bar{v}\kappa_o^{-2} v\,. \tag{2.54}$$

Furthermore, note that for any $N, \kappa_{e,}\kappa_o$ we have

$$\bar{w}\kappa_e^{-2} w = \bar{v}\kappa_o^{-2} v = \sum_{o>0} \kappa_o^{-2} - \sum_{e>0} \kappa_e^{-2}\,. \tag{2.55}$$

The right hand side converges to $\frac{\pi^2}{12}$ if we take the open string limit $\kappa_e = e,\ \kappa_o = o,\ N = \infty$. More relations of this type can be found in the next subsection.

2.1.4 GL$(N|N)$ supergroup property of the regulator

In this subsection we take a small detour to make an observation on the regulator whose significance for computations in MSFT is not yet fully apparent, but which is mathematically interesting, and could be useful in future applications. Many computations in MSFT boil down to expressions of the form $\bar{w} f(\kappa)\, w$ where $f(\kappa)$ is a matrix constructed from the frequencies κ_e, κ_o through the regulated matrices we discussed above. Therefore, we are interested in developing analytic methods of computation involving such expressions, in particular for arbitrary frequencies κ_e, κ_o. In such computations the properties of the supergroup GL$(N|N)$ mysteriously makes an appearance, as follows.

By using explicitly the expression for w_e given in eq. (2.44), we have

$$\bar{w} f\left(\kappa_e^2\right) w = \sum_e f\left(\kappa_e^2\right) \frac{\det_o\left(\kappa_e^2\kappa_o^{-2} - 1\right)}{\det_{e'\neq e}\left(\kappa_e^2\kappa_{e'}^{-2} - 1\right)} = \oint \frac{dz}{2\pi i}\frac{f(z)}{z}\frac{\det\left(1 - z\kappa_o^{-2}\right)}{\det\left(1 - z\kappa_e^{-2}\right)}\,, \tag{2.56}$$

where the contour encircles only the poles at $z = \kappa_e^2$. The contour may then be deformed to evaluate the integral. When $f\left(\kappa_e^2\right) = 1$ or κ_e^{-2} the results have already been given in eqs. (2.49), (2.55). We note that these may be written in the form

$$\bar{w} w = -1 + \mathrm{Sdet}\left(\kappa^2\right)\,, \qquad \bar{w}\kappa_e^{-2} w = -Str\left(\kappa^{-2}\right)\,, \tag{2.57}$$

where we used the superdeterminant (Sdet) and supertrace (Str) by treating the matrix

$$\kappa = \begin{pmatrix} \kappa_e & 0 \\ 0 & \kappa_o \end{pmatrix} \tag{2.58}$$

as if it is a graded GL$(N|N)$ super matrix. When $f\left(\kappa_e^2\right) = \kappa_e^{-2n}$, with $n = 1, 2, \ldots$, we note that the contour integral is precisely the integral representation of the supercharacter of GL$(N|N)$ for the representation of GL$(N|N)$ described by a Young supertableau with a single row with n superboxes [17]

$$\bar{w}\kappa_e^{-2n} w = -\chi_n\left(\kappa^{-2}\right)\,. \tag{2.59}$$

JHEP07(2003)027

The expression for the supercharacter $\chi_n(M)$ for any supermatrix M, can also be written in terms of supertraces of powers of M in the fundamental representation, as given in [17]. By taking advantage of this observation we evaluate $\chi_n\left(\kappa^{-2}\right)$ in terms of the supertraces of powers of κ^{-2}. For example, $\chi_2(M)=\frac{1}{2}Str\left(M^2\right)+\frac{1}{2}\left(Str\left(M\right)\right)^2$, which gives the following interesting expression for the sum

$$\bar{w}\kappa_e^{-4}w=-\chi_2\left(\kappa^{-2}\right)=-\frac{1}{2}\left[Tr\left(\kappa_e^{-4}\right)-Tr\left(\kappa_o^{-4}\right)\right]-\frac{1}{2}\left[Tr\left(\kappa_e^{-2}\right)-Tr\left(\kappa_o^{-2}\right)\right]^2. \quad (2.60)$$

We can check the correctness of eq. (2.59) in the limit $\kappa_e=e,\ \kappa_o=o,\ N=\infty$. In this limit, since $w_e\to\sqrt{2}\,i^{2-e}$, we can evaluate the sum on the left side directly in terms of the zeta function, $\bar{w}\kappa_e^{-2n}w\to 2\sum_{k=1}^{\infty}(2k)^{-2n}=\frac{2}{2^{2n}}\zeta(2n)$, and then compare it to the value generated by the supercharacter $-\chi_n\left(\kappa^{-2}\right)$ in the limit.

For example, for $n=1$ the right hand side of eq. (2.59) is already given following eq. (2.55), and this agrees with the left hand side which is $\frac{2}{2^2}\zeta(2)$. Similarly, for $n=2$ the left hand side of eq. (2.60) gives $\frac{2}{2^4}\zeta(4)=\frac{1}{720}\pi^4$ while the right hand side gives $\frac{7}{16}\zeta(4)-\frac{1}{8}\left(\zeta(2)\right)^2=\frac{1}{720}\pi^4$ which agree. The GL($N|N$) supergroup property of these sums is intriguing. It may be the signal of an underlying mathematical structure that could be helpful in computations in MSFT.

2.2 Moyal $\star$ from Witten's $*$ for fermionic modes

The second step in constructing the map from Witten's star to Moyal's star is to perform the Fourier transformation from position space to momentum space for a subset of string modes [7]. We recall the definition of Witten's $*$-product for functions of split strings in the ghost sector

$$\Psi_1*\Psi_2[l(\sigma),r(\sigma)]\sim\int\mathcal{D}z(\sigma)\Psi_1[l(\sigma),z(\sigma)]\Psi_2[\pm z(\sigma),r(\sigma)]. \quad (2.61)$$

For the bc ghost sector, we consider two types of overlapping conditions [2]. The anti-overlapping condition (resp. overlapping condition) is defined by choosing the minus (resp. plus) sign in this formula. In both cases the Witten $*$ product is mapped to the Moyal $\star$ product as follows.

We denote the string field as $\Psi[l,r]$ in the split string formulation and as $\hat{\Psi}[x,p]$ in the Moyal formulation. The variables l,r,x,p are all fermionic and we consider the simplified situation where each of them represents a single degree of freedom. The generalization to multiple variables is straightforward. For the simplified setup we define Witten's star product in the split string formalism (ignoring the midpoint for the time being) as

$$\Psi_1*\Psi_2[l,r]=(-1)^{|\Psi_1|}\int dw\,\Psi_1[l,w]\,\Psi_2[\pm w,r]. \quad (2.62)$$

The sign factor $(-1)^{|\Psi_1|}$ (Grassmann parity of Ψ_1) is needed to make the $*$ product associative. We define the mapping from a string field in the split string picture $\Psi[l,r]$ to the

Moyal picture $\hat{\Psi}[x,p]$ by using the Fourier transform[3]

$$\begin{aligned} \hat{\Psi}[x,p] &= \pm \int dy\, e^{-py}\, \Psi\left[\pm x + \frac{y}{2}, x \mp \frac{y}{2}\right], & (2.63)\\ \Psi[l,r] &= \pm \int dp\, e^{p(l\mp r)}\, \hat{\Psi}\left[\frac{r \pm l}{2}, p\right]. & (2.64) \end{aligned}$$

Witten's star for Ψ is then mapped to Moyal's star for $\hat{\Psi}$:

$$\hat{\Psi}_1 \star \hat{\Psi}_2[x,p] = \hat{\Psi}_1[x,p] \exp\left(\mp\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} + \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)\right)\hat{\Psi}_2[x,p]. \qquad (2.65)$$

The derivation of this correspondence is completely parallel to the bosonic case [7]:

$$\begin{aligned} &\hat{\Psi}_1[x,p] e^{\mp\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}+\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)}\hat{\Psi}_2[x,p] = \\ &= \left(\pm\int dy_1\, e^{-py_1}\,\Psi_1\left[\pm x + \frac{y_1}{2}, x \mp \frac{y_1}{2}\right]\right) e^{\mp\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}+\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)} \times \\ &\quad \times \left(\pm\int dy_2\, e^{-py_2}\,\Psi_2\left[\pm x + \frac{y_2}{2}, x \mp \frac{y_2}{2}\right]\right) \\ &= (-1)^{|\Psi_1|}\int dy_1 dy_2 \left(\Psi_1\left[\pm x + \frac{y_1}{2}, x \mp \frac{y_1}{2}\right] e^{-py_1}\right) e^{\mp\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}+\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)} \times \\ &\quad \times \left(e^{-py_2}\,\Psi_2\left[\pm x + \frac{y_2}{2}, x \mp \frac{y_2}{2}\right]\right) \\ &= (-1)^{|\Psi_1|}\int dy_1 dy_2\, \Psi_1\left[\pm x + \frac{y_1}{2}, x \mp \frac{y_1}{2}\right] e^{\pm\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}y_2} e^{-p(y_1+y_2)} e^{\mp\frac{1}{2}y_1\frac{\overrightarrow{\partial}}{\partial x}} \times \\ &\quad \times \Psi_2\left[\pm x + \frac{y_2}{2}, x \mp \frac{y_2}{2}\right] \\ &= (-1)^{|\Psi_1|}\int dy_1 dy_2\, e^{-p(y_1+y_2)}\Psi_1\left[\pm x + \frac{y_1+y_2}{2}, x \mp \frac{y_1-y_2}{2}\right] \times \\ &\quad \times \Psi_2\left[\pm x - \frac{y_1-y_2}{2}, x \mp \frac{y_1+y_2}{2}\right] \\ &= \pm(-1)^{|\Psi_1|}\int dy e^{-py}\int dz\, \Psi_1\left[\pm x + \frac{y}{2}, z\right]\Psi_2\left[\pm z, x \mp \frac{y}{2}\right] \\ &= \pm\int dy e^{-py}\, (\Psi_1 * \Psi_2)\left[\pm x + \frac{y}{2}, x \mp \frac{y}{2}\right] =: \hat{\Psi}_1 \star \hat{\Psi}_2[x,p]. \qquad (2.66) \end{aligned}$$

The form of the product in eq. (2.65) is similar to the ordinary Moyal product although the derivatives in the exponential are for fermionic variables. This Moyal $\star$ product is associative and non-commutative.

From eq. (2.64), we also obtain the correspondence between the definition of trace in the split-string formulation, which we take with the anti-periodic condition, and that of Moyal one which is given by an integration in "phase space"

$$\mathrm{Tr}\,\Psi := \int dz\, \Psi[\pm z, z] = \pm\int dxdp\, \hat{\Psi}[x,p] =: \pm\mathrm{Tr}\,\hat{\Psi}. \qquad (2.67)$$

[3] If l and r consist of N variables, the sign factor on the right hand sides of eqs. (2.62) (2.63) (2.64) become $(-1)^{|\Psi_1|N}, (\pm 1)^N, (\pm 1)^N$ respectively. In particular, they are trivial in the case that N is even.

JHEP07(2003)027

2.3 Moyal $\star$ product in the bc ghost sector

In this section we define the Moyal $\star$ product which represents Witten's star product using the results in section 2.1, section 2.2, appendix A.1. We first review the conventional operator formalism to fix the notation in the bc ghost sector. We take the ghost coordinates $b(\sigma), c(\sigma)$ and their conjugates $\pi_b(\sigma), \pi_c(\sigma)$

$$b^{\pm}(\sigma) = \sum_{n=-\infty}^{\infty} \hat{b}_n e^{\pm in\sigma} = \pi_c(\sigma) \mp ib(\sigma) \,, \qquad c^{\pm}(\sigma) = \sum_{n=-\infty}^{\infty} \hat{c}_n e^{\pm in\sigma} = c(\sigma) \pm i\pi_b(\sigma) \,. \tag{2.68}$$

We introduce the fermionic variables $\hat{x}_n, \hat{y}_n$ and their conjugates $\hat{p}_n, \hat{q}_n$ as follows

$$b(\sigma) \;=\; -\sum_{n=1}^{\infty}(\hat{b}_n - \hat{b}_{-n}) \sin n\sigma = i\sqrt{2}\sum_{n=1}^{\infty} \hat{x}_n \sin n\sigma \,, \tag{2.69}$$

$$c(\sigma) \;=\; \hat{c}_0 + \sum_{n=1}^{\infty}(\hat{c}_n + \hat{c}_{-n}) \cos n\sigma = \hat{c}_0 + \sqrt{2}\sum_{n=1}^{\infty} \hat{y}_n \cos n\sigma \,, \tag{2.70}$$

$$\pi_b(\sigma) \;=\; \sum_{n=1}^{\infty}(\hat{c}_n - \hat{c}_{-n}) \sin n\sigma = -i\sqrt{2}\sum_{n=1}^{\infty} \hat{p}_n \sin n\sigma \,, \tag{2.71}$$

$$\pi_c(\sigma) \;=\; \hat{b}_0 + \sum_{n=1}^{\infty}(\hat{b}_n + \hat{b}_{-n}) \cos n\sigma = \hat{b}_0 + \sqrt{2}\sum_{n=1}^{\infty} \hat{q}_n \cos n\sigma \,. \tag{2.72}$$

The nonzero modes $\hat{x}_n, \hat{y}_n, \hat{p}_n, \hat{q}_n$ are related to $\hat{b}_n, \hat{c}_n$:

$$\hat{x}_n = \frac{i}{\sqrt{2}}(\hat{b}_n - \hat{b}_{-n}) \,, \quad \hat{y}_n = \frac{1}{\sqrt{2}}(\hat{c}_n + \hat{c}_{-n}) \,, \quad \hat{p}_n = \frac{i}{\sqrt{2}}(\hat{c}_n - \hat{c}_{-n}) \,, \quad \hat{q}_n = \frac{1}{\sqrt{2}}(\hat{b}_n + \hat{b}_{-n}) \,, \tag{2.73}$$

and the canonical commutation relation $\{\hat{b}_n, \hat{c}_m\} = \delta_{m+n,0}$ can be rewritten as

$$\{\hat{x}_n, \hat{p}_m\} = \delta_{n,m} \,, \qquad \{\hat{y}_n, \hat{q}_m\} = \delta_{n,m} \,, \qquad n, m = 1, 2, \ldots \,. \tag{2.74}$$

We represent the string field by treating c_0 and x_n, y_n as the "position" coordinates. The translation between Fock space representation and position representation is made through

$$\Psi(c_0, x_n, y_n) = \langle c_0, x_n, y_n | \Psi \rangle \,. \tag{2.75}$$

Here we introduced the bra state $\langle c_0, x_n, y_n|$ (and the corresponding ket state) as states in Fock space which satisfy the eigenvalue conditions for the operators $\hat{c}_0, \hat{x}_n, \hat{y}_n$,

$$\begin{aligned}
\langle c_0, x_n, y_n | \hat{c}_0 \;&=\; \langle c_0, x_n, y_n | c_0 \,, \\
\langle c_0, x_n, y_n | \hat{x}_n \;&=\; \langle c_0, x_n, y_n | x_n \,, \\
\langle c_0, x_n, y_n | \hat{y}_n \;&=\; \langle c_0, x_n, y_n | y_n \,, \\
\hat{c}_0 | c_0, x_n, y_n \rangle \;&=\; c_0 | c_0, x_n, y_n \rangle \,, \\
\hat{x}_n | c_0, x_n, y_n \rangle \;&=\; x_n | c_0, x_n, y_n \rangle \,, \\
\hat{y}_n | c_0, x_n, y_n \rangle \;&=\; y_n | c_0, x_n, y_n \rangle \,.
\end{aligned}$$

Explicitly these are given by

$$\langle c_0, x_n, y_n| = \langle\Omega|\hat{c}_{-1}\hat{c}_0 \exp\left(c_0\hat{b}_0+\sum_{n=1}^{\infty}\left(-\hat{c}_n\hat{b}_n - i\sqrt{2}\hat{c}_n x_n+\sqrt{2}y_n\hat{b}_n+iy_n x_n\right)\right), \quad (2.76)$$

$$|c_0, x_n, y_n\rangle = \exp\left(\hat{b}_0 c_0+\sum_{n=1}^{\infty}\left(-\hat{b}_{-n}\hat{c}_{-n}+i\sqrt{2}x_n\hat{c}_{-n}+\sqrt{2}\hat{b}_{-n}y_n-ix_n y_n\right)\right)\hat{c}_0\hat{c}_1|\Omega\rangle \quad (2.77)$$

where $\langle\Omega|, |\Omega\rangle$ represents the conformal vacuum[4] normalized as $\langle\Omega|\hat{c}_{-1}\hat{c}_0\hat{c}_1|\Omega\rangle = 1$. These bras and kets satisfy the normalization and completeness relations

$$\langle c_0, x_n, y_n|c_0', x_n', y_n'\rangle = -(c_0-c_0')\prod_{n=1}^{\infty}\left(-2i(x_n-x_n')(y_n-y_n')\right), \quad (2.78)$$

$$-\int dc_0\int\prod_{n=1}^{\infty}\frac{dx_n dy_n}{2i}|c_0, x_n, y_n\rangle\langle c_0, x_n, y_n| = 1\,. \quad (2.79)$$

Witten's star product for the ghost sector is defined by the (anti-)overlapping conditions [2],

$$b^{\pm(r)}(\sigma) - b^{\pm(r-1)}(\pi-\sigma) = 0\,, \qquad c^{\pm(r)}(\sigma) + c^{\pm(r-1)}(\pi-\sigma) = 0\,, \quad (2.80)$$

for $r = 1, 2, 3 \bmod 3$, $\sigma \in [0, \pi/2]$, or equivalently

$$b^{(r)}(\sigma) - b^{(r-1)}(\pi-\sigma) = 0\,, \qquad c^{(r)}(\sigma) + c^{(r-1)}(\pi-\sigma) = 0\,, \quad (2.81)$$

$$\pi_b^{(r)}(\sigma) + \pi_b^{(r-1)}(\pi-\sigma) = 0\,, \qquad \pi_c^{(r)}(\sigma) - \pi_c^{(r-1)}(\pi-\sigma) = 0\,. \quad (2.82)$$

These (anti-)overlapping conditions for bc ghost will be used to define the mapping from Witten's $*$ to Moyal's $\star$ by using eq.(2.63) defined in the previous section.

To apply the formulation in section 2.2, we need to specify the boundary conditions of the split string variables, since we have to use the Bogoliubov transformation given in section 2.1/appendix A.1 accordingly. At $\sigma = 0, \pi$, $b(\sigma)$ (resp. $c(\sigma)$) satisfies the Dirichlet (resp. Neumann) boundary condition. On the other hand, at the midpoint, there are two options, namely Neumann or Dirichlet type boundary conditions. In the following we choose the Neumann condition for $b(\sigma)$ and Dirichlet condition for $c(\sigma)$.[5]

In the split string language, the left and right halves of $b(\sigma)$, $l^b(\sigma), r^b(\sigma)$, satisfy Dirichlet at $\sigma = 0$ and Neumann at $\sigma = \pi/2$, while the left and right halves of $c(\sigma)$, $l^c(\sigma), r^c(\sigma)$, satisfy Neumann at $\sigma = 0$ and Dirichlet at $\pi/2$. With this choice, $l^b(\sigma), r^b(\sigma)$ are expanded by using *odd* sine modes: $\{\sin o\sigma,\ o = 1, 3, 5, \ldots\}$, and $l^c(\sigma), r^c(\sigma)$ by using *odd* cosine modes: $\{\cos o\sigma,\ o = 1, 3, 5, \ldots\}$:

$$l^b(\sigma) = i\sqrt{2}\sum_{o=1}^{\infty} l_o^b \sin o\sigma\,, \qquad r^b(\sigma) = i\sqrt{2}\sum_{o=1}^{\infty} r_o^b \sin o\sigma\,, \quad (2.83)$$

[4] We take the convention that $|\Omega\rangle$ is Grassmann even and $\langle\Omega|$ is odd.

[5] The other choice (Neumann for b and Dirichlet for c at the midpoint) is discussed in the appendix D. It gives equivalent but more complicated expression for the Moyal formulation.

$$l^c(\sigma) = \bar{c} + \sqrt{2}\sum_{o=1}^{\infty} l_o^c \cos o\sigma\,, \qquad r^c(\sigma) = \bar{c} + \sqrt{2}\sum_{o=1}^{\infty} r_o^c \cos o\sigma\,. \tag{2.84}$$

From eqs. (2.69) (2.70) (2.18) (A.7), we have the relations between split- and full-string variables

$$l_o^b = \bar{S}x_e + x_o\,, \qquad r_o^b = -\bar{S}x_e + x_o\,, \tag{2.85}$$

$$\bar{c} = c_0 - \bar{w}y_e\,, \qquad l_o^c = Ry_e + y_o\,, \qquad r_o^c = Ry_e - y_o \tag{2.86}$$

where we used a matrix notation. Witten's $*$ product for the split string formulation is written as[6]

$$\tilde{A} * \tilde{B}(\bar{c}, l_o^b, l_o^c, r_o^b, r_o^c) = \int \prod_{o>0} \left(i d\eta_o^b d\eta_o^c\right) \tilde{A}\left(\bar{c}, l_o^b, l_o^c, \eta_o^b, \eta_o^c\right) \tilde{B}\left(\bar{c}, \eta_o^b, -\eta_o^c, r_o^b, r_o^c\right). \tag{2.87}$$

The string field in the split-string formulation is identified with the usual position representation $\Psi(x(\sigma))$, which is written in terms of modes

$$\tilde{A}(\bar{c}, l_o^b, l_o^c, r_o^b, r_o^c) \sim \Psi(c_0, x_n, y_n) := \langle c_0, x_n, y_n | \Psi\rangle\,. \tag{2.88}$$

In order to map it to the Moyal formulation, we compare eqs. (2.85) (2.86) with eq. (2.63), and note the similarities (we add prime $'$ to distinguish the variables with anti-overlapping condition from the variables with overlapping conditions)

$$\bar{S}x_e + x_o \sim x + \frac{y}{2}\,, \qquad -\bar{S}x_e + x_o \sim x - \frac{y}{2}\,, \qquad Ry_e + y_o \sim -x' + \frac{y'}{2}\,, \qquad Ry_e - y_o \sim x' + \frac{y'}{2} \tag{2.89}$$

or equivalently[7]

$$x_o \sim x\,, \qquad \bar{S}x_e \sim \frac{y}{2}\,, \qquad y_o \sim -x'\,, \qquad Ry_e \sim \frac{y'}{2}\,. \tag{2.90}$$

Thus, using eqs. (2.63) (2.88), we obtain the map from the field in the position representation to the Moyal representation

$$A(\bar{c}, x_o, p_o, y_o, q_o) := 2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}} \int \prod_{e>0} \left(i^{-1} dx_e dy_e\right) e^{-2p_o\bar{S}x_e - 2q_o R y_e}\, \Psi(\bar{c}+\bar{w}y_e, x_n, y_n)\,. \tag{2.91}$$

At this point, we used the MSFT regularization scheme, by truncating the ghost modes x_n, y_n to $n \leq 2N$, and using the parameters (N, κ_e, κ_o) given in the previous section. Thus, w, R, S are redefined in eqs. (A.17) (A.16) (2.50) and $2^{2N}, \bar{w}w$ are finite. We fixed the

[6]Here we consider *naive* overlapping condition. To obtain the conventional Witten's star product, as we will show, we should treat midpoint variable $\bar{c}$ more carefully. The phase factor i in the measure is only convention so that it is "real" $(idx_e dy_e)^\dagger = idx_e dy_e$. Here we define complex conjugate for fermionic variables ξ, ξ' as $(\xi\xi')^\dagger = (\xi')^\dagger(\xi)^\dagger$.

[7]We note that the *odd* modes correspond to the "x-variable" of phase space in the Moyal formulation of ghosts. By contrast, in the matter sector, the even modes played the corresponding rôle (see eq.(30) in [7]).

normalization factor $\left(\det(16\bar{S}\bar{R})\right)^{-\frac{1}{2}} = 2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}$ consistently with the trace that will be given later. Hence, from eqs.(2.65) (2.90), the Moyal $\star$ product that corresponds to Witten's $*$ product is

$$
\begin{aligned}
A \star B(\bar{c}, x_o, p_o, y_o, q_o) = {} & A(\bar{c}, x_o, p_o, y_o, q_o)\, e^{-\frac{1}{2}\sum_{o>0}\left(\frac{\overleftarrow{\partial}}{\partial x_o}\frac{\overrightarrow{\partial}}{\partial p_o}+\frac{\overleftarrow{\partial}}{\partial y_o}\frac{\overrightarrow{\partial}}{\partial q_o}+\frac{\overleftarrow{\partial}}{\partial p_o}\frac{\overrightarrow{\partial}}{\partial x_o}+\frac{\overleftarrow{\partial}}{\partial q_o}\frac{\overrightarrow{\partial}}{\partial y_o}\right)} \times \\
& \times B(\bar{c}, x_o, p_o, y_o, q_o)\,.
\end{aligned}
\tag{2.92}
$$

We introduced an arbitrary parameter θ' which is the analog of θ in the matter sector [9] to absorb units. We note that the product is local as a function of the midpoint $\bar{c}$, while $\bar{c}$ is related to the center or mass variable c_0 by

$$
c_0 = \bar{c} + \bar{w} y_e\,. \tag{2.93}
$$

By rescaling variables and performing Fourier transformation with respect to $\bar{c}$, we arrive at the definition of the string field in MSFT

$$
\begin{aligned}
\hat{A}(\xi_0, x_o, p_o, y_o, q_o) &= \int d\bar{c} e^{-\xi_0 \bar{c}} A\left(\bar{c}, x_o, -\frac{p_o}{\theta'}, y_o, -\frac{q_o}{\theta'}\right) \\
&= 2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}} \int dc_0 \prod_{e>0}\left(i^{-1} dx_e dy_e\right) \times \\
&\quad \times e^{-\xi_0 c_0 + \xi_0 \bar{w} y_e + \frac{2}{\theta'} p_o \bar{S} x_e + \frac{2}{\theta'} q_o R y_e}\, \Psi(c_0, x_n, y_n)\,.
\end{aligned}
\tag{2.94}
$$

By this Fourier transformation with respect to zero mode, the Grassmann parity of $\hat{A}(\xi_0, x_o, p_o, y_o, q_o)$ and the corresponding $|\Psi\rangle$ coincide. The Moyal $\star$ product which is modified after eqs. (2.92) (2.94) is

$$
\star := \exp\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial \xi}\Sigma\frac{\overrightarrow{\partial}}{\partial \xi}\right) = \exp\left(\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial \xi_1}\sigma'\frac{\overrightarrow{\partial}}{\partial \xi_1} + \frac{\overleftarrow{\partial}}{\partial \xi_2}\sigma'\frac{\overrightarrow{\partial}}{\partial \xi_2}\right)\right), \tag{2.95}
$$

$$
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} x_o \\ p_o \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} y_o \\ q_o \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma' \end{pmatrix}, \quad \sigma' = \theta' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.96}
$$

We define the trace in MSFT as integration over the "phase space":

$$
\mathrm{Tr}\, \hat{A}(\xi_0, \xi) = \det \sigma' \int d\xi\, \hat{A}(\xi_0, \xi) = (-1)^N \theta'^{2N} \int \prod_{o>0} (dx_o dp_o dy_o dq_o)\, \hat{A}(\xi_0, \xi)\,. \tag{2.97}
$$

The Moyal field $\hat{A}(\xi_0, \xi)$ which is mapped from $|\Psi\rangle$ by (2.94) is normalized as

$$
\int d\xi_0 \mathrm{Tr}\left(\left(\hat{A}(\xi_0, \xi)\right)^{\dagger} \star \left(\frac{\partial}{\partial \xi_0} - \frac{\theta'}{2}\bar{v}\frac{\partial}{\partial q_o}\right)\hat{A}(\xi_0, \xi)\right) = \langle \Psi | \hat{c}_0 | \Psi \rangle\,, \tag{2.98}
$$

where $\frac{\partial}{\partial \xi_0} - \frac{\theta'}{2}\bar{v}\frac{\partial}{\partial q_o}$ corresponds to $-\hat{c}_0$ (2.107).

It is convenient to introduce the bra $\langle \xi_0, \xi|$ as a state in Fock space such that the Moyal field is related directly to the Fock space field via $\hat{A}(\xi_0, \xi) = \langle \xi_0, \xi | \Psi \rangle$. This can be obtained

JHEP07(2003)027

from the bra state $\langle c_0, x_n, y_n|$ (2.77) by Fourier transformation (2.94),

$$\begin{aligned}\langle\xi_0,\xi| &= \langle\xi_0,\xi_1,\xi_2| = \langle\xi_0,x_o,p_o,y_o,q_o| \\ &= 2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}\int dc_0\prod_{e>0}\left(i^{-1}dx_e dy_e\right)e^{-\xi_0c_0+\xi_0\bar{w}y_e+\frac{2}{\theta'}p_o\bar{S}x_e+\frac{2}{\theta'}q_oRy_e}\langle c_0,x_n,y_n| \\ &= -2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}\langle\Omega|\hat{c}_{-1}e^{-\xi_0(\hat{c}_0-\sqrt{2}\bar{w}\hat{c}_e)}e^{\hat{c}_e\hat{b}_e-\hat{c}_o\hat{b}_o-2i\bar{\xi}_1M_0^{(o)}\xi_2-\bar{\xi}_1\lambda_1-\bar{\xi}_2\lambda_2}\end{aligned} \tag{2.99}$$

where we used notation:

$$M_0^{(o)} = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{2}{\theta'^2}\bar{S}\bar{R}\end{pmatrix}, \qquad \lambda_1 = \begin{pmatrix}-i\sqrt{2}\hat{c}_o\\ -\frac{2\sqrt{2}i}{\theta'}\bar{S}\hat{b}_e+\frac{2i}{\theta'}\bar{S}w\xi_0\end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix}-\sqrt{2}\hat{b}_o\\ -\frac{2\sqrt{2}}{\theta'}R\hat{c}_e\end{pmatrix}. \tag{2.100}$$

This is the result given in the summary at the beginning of this section.

Examples. Here we give some examples of string fields in MSFT.

For the conventional ghost number 1 vacuum $\hat{c}_1|\Omega\rangle$ and SL(2, R) invariant vacuum $|\Omega\rangle$, the corresponding fields are given by

$$\hat{A}_0(\xi_0,\xi) = \langle\xi_0,\xi|\hat{c}_1|\Omega\rangle = 2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}\xi_0\, e^{-ix_oy_o-i\frac{4}{\theta'^2}p_o(\bar{S}\bar{R})_{oo'}q_{o'}}, \tag{2.101}$$

$$\hat{A}_\Omega(\xi_0,\xi) = \langle\xi_0,\xi|\Omega\rangle = -i\,2^{-2N+\frac{1}{2}}(1+\bar{w}w)^{-\frac{1}{4}}\,\xi_0\,x_1\,e^{-ix_oy_o-i4p_o(\bar{S}\bar{R})_{oo'}q_{o'}}. \tag{2.102}$$

We note that in the open string limit $\kappa_e = e$, $\kappa_o = o$, $N=\infty$, these expressions become singular. For example, the coefficient of $p_oq_{o'}$ is divergent,

$$(\bar{S}\bar{R})_{oo'} = \frac{16\,i^{o+o'+2}}{\pi^2 o'}\sum_{e=2}^{\infty}\frac{e^3}{(e^2-(o')^2)(e^2-o^2)} = \pm\infty\,. \tag{2.103}$$

Therefore, it is advisable not to take the open string limit at the level of the state, but wait for the end of a computation. We note that the physical quantities such as the scattering amplitude become regular in this limit [10].

For the identity-like state in the Siegel gauge: $|\tilde{I}\rangle = \mathcal{N}_{\tilde{I}}\,e^{\sum_{n=1}^{\infty}(-1)^n\hat{c}_{-n}\hat{b}_{-n}}\hat{c}_1|\Omega\rangle$, which is a delta function with respect to the even modes in position basis, $\Psi_{\tilde{I}}(c_0,x_n,y_n) = \mathcal{N}_{\tilde{I}}\prod_{e>0}(-4i\delta(x_e)\delta(y_e))$, the corresponding field is

$$\hat{A}_{\tilde{I}}(\xi_0,x_o,p_o,y_o,q_o) = (-1)^N(1+\bar{w}w)^{-\frac{1}{4}}\mathcal{N}_{\tilde{I}}\,\xi_0\,.$$

Except for the zero mode and the normalization factor, this $\hat{A}_{\tilde{I}}$ is the identity element with respect to the Moyal $\star$ product. The conventional identity state $|I\rangle$ (C.1) [2] (which is BRST invariant, and not in the Siegel gauge) becomes more complicated in MSFT.

2.4 Oscillators

In this subsection, we obtain the Moyal images of the conventional oscillators which are used in applications in MSFT. In this way we can write various operators in oscillator language and in particular discuss the form of L_0 and the butterfly state which came up in our work in [12].

2.4.1 Oscillators on the fields in MSFT

For an operator $\hat{\mathcal{O}}$ which consists of $\hat{b}_n, \hat{c}_n$, acting on a state $|\Psi\rangle$ in Fock space, we define its Moyal image $\hat{\beta}_{\hat{\mathcal{O}}}$, which is a differential operator acting on the Moyal field $\hat{A}_\Psi(\xi_0,\xi) = \langle \xi_0, \xi|\Psi\rangle$, as follows

$$\hat{\beta}_{\hat{\mathcal{O}}} \hat{A}_\Psi(\xi_0,\xi) = \langle \xi_0, \xi|\hat{\mathcal{O}}|\Psi\rangle \,. \tag{2.104}$$

For the basic operators, $\hat{c}_0, \hat{b}_0, \hat{x}_o, \hat{y}_o, \hat{x}_e, \hat{y}_e, \hat{p}_o, \hat{q}_o, \hat{p}_e, \hat{q}_e$, this rule gives the corresponding operators in MSFT

$$\hat{\beta}_{\hat{c}_0} = -\frac{\partial}{\partial \xi_0} + \frac{\theta'}{2}\bar{v}\frac{\partial}{\partial q_o}\,, \qquad \hat{\beta}_{\hat{b}_0} = -\xi_0\,, \tag{2.105}$$

$$\hat{\beta}_{\hat{x}_o} = x_o\,, \qquad \hat{\beta}_{\hat{y}_o} = y_o\,, \qquad \hat{\beta}_{\hat{x}_e} = \frac{\theta'}{2} S \frac{\partial}{\partial p_o}\,, \qquad \beta_{\hat{y}_e} = \frac{\theta'}{2} T \frac{\partial}{\partial q_o}\,, \tag{2.106}$$

$$\hat{\beta}_{\hat{p}_o} = \frac{\partial}{\partial x_o}\,, \qquad \hat{\beta}_{\hat{q}_o} = \frac{\partial}{\partial y_o}\,, \qquad \hat{\beta}_{\hat{p}_e} = \frac{2}{\theta'} S p_o\,, \qquad \hat{\beta}_{\hat{q}_e} = \frac{2}{\theta'} \bar{R}\, q_o + w_e \xi_0\,. \tag{2.107}$$

The nonzero modes of the oscillators $\hat{b}_n, \hat{c}_n$ become

$$\hat{\beta}^b_e = \frac{1}{\sqrt{2}} \sum_{o>0} \left(\frac{2}{\theta'} q_o R_{o|e|} - i\epsilon(e) \frac{\theta'}{2} S_{|e|o} \frac{\partial}{\partial p_o} \right) + \frac{1}{\sqrt{2}} w_{|e|} \xi_0 = \sum_o \bar{\beta}^b_o U^{-1}_{-o,e} + w'_e \xi_0\,, \tag{2.108}$$

$$\hat{\beta}^b_o = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial y_{|o|}} - i\epsilon(o) x_{|o|} \right), \tag{2.109}$$

$$\hat{\beta}^c_e = \frac{1}{\sqrt{2}} \sum_{o>0} \left(\frac{\theta'}{2} T_{|e|o} \frac{\partial}{\partial q_o} - i\epsilon(e) \frac{2}{\theta'} S_{|e|o}\, p_o \right) = \sum_o U_{e,-o} \bar{\beta}^c_o\,, \tag{2.110}$$

$$\hat{\beta}^c_o = \frac{1}{\sqrt{2}} \left(y_{|o|} - i\epsilon(o) \frac{\partial}{\partial x_{|o|}} \right). \tag{2.111}$$

In the first and third equations we introduced the symbols $\bar{\beta}^b_o$,$\bar{\beta}^c_o$ to denote the differential operators

$$\bar{\beta}^b_o = \frac{1}{\sqrt{2}} \left(\frac{2}{\theta'} q_{|o|} - i\frac{\theta'}{2} \epsilon(o) \frac{\partial}{\partial p_{|o|}} \right), \qquad \bar{\beta}^c_o = \frac{1}{\sqrt{2}} \left(\frac{\theta'}{2} \frac{\partial}{\partial q_{|o|}} - i\epsilon(o) \frac{2}{\theta'} p_{|o|} \right). \tag{2.112}$$

The even and the odd sets satisfy canonical anti-commutation relations

$$\{\hat{\beta}^b_e, \hat{\beta}^c_{e'}\} = \delta_{e+e'}\,, \qquad \{\hat{\beta}^b_o, \hat{\beta}^c_{o'}\} = \delta_{o+o'}\,, \qquad \{\bar{\beta}^b_o, \bar{\beta}^c_{o'}\} = \delta_{o+o'}\,. \tag{2.113}$$

Using the above maps of operators, we can translate operators in the usual oscillator representation into MSFT language. For example, the ghost number operator (we assigned ghost number 1 to $\hat{c}_1|\Omega\rangle$)

$$N_{\rm gh} = \sum_{n\geq 1} (\hat{c}_{-n}\hat{b}_n - \hat{b}_{-n}\hat{c}_n) + \hat{c}_0\hat{b}_0 + 1\,, \tag{2.114}$$

is mapped to its MSFT image

$$N_{\rm gh} = \sum_{o>0} \left(y_o \frac{\partial}{\partial y_o} - x_o \frac{\partial}{\partial x_o} + p_o \frac{\partial}{\partial p_o} - q_o \frac{\partial}{\partial q_o} \right) - \xi_0 \frac{\partial}{\partial \xi_0} + 2\,. \tag{2.115}$$

2.4.2 Oscillator as a field

It is often useful to rewrite differential operators in Moyal space in terms of the star product. The idea is to replace the derivative by the $\star$ -(super)commutator:[8]

$$\frac{\partial}{\partial \xi}\hat{A} = \Sigma^{-1}[\xi, \hat{A}\}_\star . \tag{2.116}$$

More concretely

$$\frac{\partial}{\partial x_o}\hat{A} = \frac{1}{\theta'}[p_o, \hat{A}\}_\star , \quad \frac{\partial}{\partial y_o}\hat{A} = \frac{1}{\theta'}[q_o, \hat{A}\}_\star , \quad \frac{\partial}{\partial p_o}\hat{A} = \frac{1}{\theta'}[x_o, \hat{A}\}_\star , \quad \frac{\partial}{\partial q_o}\hat{A} = \frac{1}{\theta'}[y_o, \hat{A}\}_\star . \tag{2.117}$$

This observation leads to the star product representation of the differential operators as follows

$$\hat{\beta}^b_o\hat{A} = \frac{1}{\sqrt{2}}\left(\beta^b_o \star \hat{A} - (-1)^{|A|}\hat{A}\star\beta^b_{-o}\right), \qquad \bar{\beta}^b_o\hat{A} = \frac{1}{\sqrt{2}}\left(\beta^b_o \star \hat{A} + (-1)^{|A|}\hat{A}\star\beta^b_{-o}\right), \tag{2.118}$$

$$\hat{\beta}^c_o\hat{A} = \frac{1}{\sqrt{2}}\left(\beta^c_o \star \hat{A} + (-1)^{|A|}\hat{A}\star\beta^c_{-o}\right), \qquad \bar{\beta}^c_o\hat{A} = \frac{1}{\sqrt{2}}\left(\beta^c_o \star \hat{A} - (-1)^{|A|}\hat{A}\star\beta^c_{-o}\right), \tag{2.119}$$

where we defined the fields$\beta^{b,c}_o$ in Moyal space that play the fundamental role of oscillators

$$\beta^b_o := \frac{1}{\theta'}q_{|o|} - \frac{i}{2}\epsilon(o)x_{|o|} , \qquad \beta^c_o := \frac{1}{2}y_{|o|} - \frac{i}{\theta'}\epsilon(o)p_{|o|} . \tag{2.120}$$

The odd $\bar{\beta}^{b,c}_o$ and even $\hat{\beta}^{b,c}_e$ differential operators are related to each other as in eqs. (2.108), (2.110). Therefore we also define fields with even labels via the Bogoliubov transformation

$$\beta^b_e := \sum_o \beta^b_o U^{-1}_{-o,e} , \qquad \beta^c_e := \sum_o U_{e,-o}\beta^c_o \tag{2.121}$$

where the sum runs over odd integers from $-2N+1$ to $2N-1$. These give the star product representation of the even differential operators $\hat{\beta}^{b,c}_e$

$$\hat{\beta}^b_e\hat{A} = \frac{1}{\sqrt{2}}\left(\beta^b_e \star \hat{A} + (-1)^{|A|}\hat{A}\star\beta^b_{-e}\right) + w'_e\xi_0\hat{A} , \tag{2.122}$$

$$\hat{\beta}^c_e\hat{A} = \frac{1}{\sqrt{2}}\left(\beta^c_e \star \hat{A} - (-1)^{|A|}\hat{A}\star\beta^c_{-e}\right) . \tag{2.123}$$

The fields $\beta^{b,c}_e$ or $\beta^{b,c}_o$ satisfy the oscillator anticommutation relations with respect to the star product

$$\{\beta^b_o, \beta^c_{o'}\}_\star = \delta_{o+o'} , \qquad \{\beta^b_e, \beta^c_{e'}\}_\star = \delta_{e+e'} . \tag{2.124}$$

But they do not anticommute with each other

$$\{\beta^b_{-e}, \beta^c_o\}_\star = U^{-1}_{o,-e} , \qquad \{\beta^b_o, \beta^c_{-e}\}_\star = U_{-e,o} \tag{2.125}$$

since they are related to each other by the Bogoliubov transformation given above.

---

[8] We define the supercommutator as $[\hat{A}_1, \hat{A}_2\}_\star := \hat{A}_1\star\hat{A}_2 - (-1)^{|A_1||A_2|}\hat{A}_2\star\hat{A}_1$. Note that $\hat{A}\frac{\overleftarrow{\partial}}{\partial\xi} = -(-1)^{|A|}\frac{\overrightarrow{\partial}}{\partial\xi}\hat{A}$.

One may regard the fields $\beta^{b,c}$ as the harmonic oscillators in Moyal space which act on the string field A from either side by the star product. It is then natural to introduce the vacuum field associated with the odd (and similarly the even) oscillators, as the field that satisfies the following conditions under the star product

$$\beta_o^b \star \hat{A}_B = \beta_o^c \star \hat{A}_B = \hat{A}_B \star \beta_{-o}^b = \hat{A}_B \star \beta_{-o}^c = 0\,, \quad \forall o > 0\,. \tag{2.126}$$

By definition, this field is the Moyal image of the Fock space operator $\hat{A}_B \sim |0\rangle\langle 0|$, where $|0\rangle$ is the vacuum state with respect to the oscillators $\beta_o^{b,c}$. These are first order differential equations whose solution is the gaussian

$$\hat{A}_B = \xi_0\, 2^{-2N} \exp\left(-\sum_{o>0}\left(ix_o y_o + \frac{4i}{\theta'^2} p_o q_o\right)\right)\,. \tag{2.127}$$

If we write $\hat{A}_B = \xi_0 A_B$, we see that A_B is a projector (3.24) with respect to the Moyal $\star$ product: $A_B \star A_B = A_B$. It turns out this is the butterfly projector that came up in other formulations of string field theory [25, 26, 27] as shown in appendix E.

2.4.3 L_0 and $\mathcal{L}_0$

In string field theory computations the zeroth Virasoro operator L_0 plays a critical role since it defines the propagator. In this section, we derive various forms of L_0 in MSFT. In the usual oscillator representation acting on Fock space, L_0 is given by[9]

$$L_0 = \sum_{k=1}^{\infty} k(\hat{b}_{-k}\hat{c}_k + \hat{c}_{-k}\hat{b}_k)\,. \tag{2.128}$$

In MSFT which is regularized by (N, κ_e, κ_o), we truncate the number of oscillators to $2N$ and replace the frequencies by $\kappa_{e,o}$. Then the Moyal image of L_0 becomes the following differential operator

$$\begin{aligned} L_0 &= \sum_{k=1}^{2N} \kappa_k(\hat{\beta}_{-k}^b \hat{\beta}_k^c + \hat{\beta}_{-k}^c \hat{\beta}_k^b) && (2.129)\\ &= \sum_{k=1}^{2N} \kappa_k + i\sum_{o>0} \kappa_o \left(x_o y_o + \frac{\partial}{\partial x_o}\frac{\partial}{\partial y_o} + \frac{4}{\theta'^2} p_o q_o + \frac{\theta'^2}{4}\frac{\partial}{\partial p_o}\frac{\partial}{\partial q_o}\right) + \\ &\quad + \frac{4i}{\theta'^2}(1+\bar{w}w)\left(\sum_{o>0}\kappa_o v_o p_o\right)\left(\sum_{o'>0} v_{o'} q_{o'}\right) + \frac{2i}{\theta'}(1+\bar{w}w)\left(\sum_{o>0} v_o \kappa_o p_o\right)\xi_0\,. && (2.130) \end{aligned}$$

The last two terms come from the identities

$$\bar{S}\kappa_e \bar{R} = \kappa_o R\bar{R} = \kappa_o + (1+\bar{w}w)\kappa_o v\bar{v}\,, \qquad \sum_{e>0} S_{eo}\kappa_e w_e = \kappa_o R w_e = \kappa_o v(1+\bar{w}w)\,. \tag{2.131}$$

[9]We take the convention such that $L_0(\hat{c}_1|\Omega\rangle) = 0$ fixes the constant that comes from normal ordering.

The similarity to the matter sector is enhanced by introducing an even basis $\left(x_e^b, p_e^b, x_e^c, p_e^c\right)$ which is related to the odd basis through the following linear canonical transformation[10]

$$x_e^b := \kappa_e^{-1} S x_o\,, \qquad p_e^b := \kappa_e S p_o\,, \qquad x_e^c := T y_o\,, \qquad p_e^c := \bar{R} q_o\,. \tag{2.132}$$

The Moyal $\star$ product (2.95) and trace (2.97) are invariant under this canonical transformation. With the new variables, L_0 is rewritten as

$$\begin{aligned} L_0 =& \sum_{k=1}^{2N} \kappa_k + i \sum_{e>0} \left(\kappa_e^2\, x_e^b x_e^c + \frac{\partial}{\partial x_e^b}\frac{\partial}{\partial x_e^c} + \frac{4}{\theta'^2} p_e^b p_e^c + \frac{\theta'^2}{4}\kappa_e^2 \frac{\partial}{\partial p_e^b}\frac{\partial}{\partial p_e^c}\right) - \\ & - \frac{i}{1+\bar{w}w} \left(\sum_{e>0} w_e \frac{\partial}{\partial x_e^b}\right)\left(\sum_{e'>0} w_{e'} \frac{\partial}{\partial x_{e'}^c}\right) + \frac{2i}{\theta'}\left(\sum_{e>0} w_e p_e^b\right)\xi_0\,. \end{aligned} \tag{2.133}$$

Under this change of variables, the usual perturbative vacuum that was given in the odd basis (2.101) becomes:

$$\hat{A}_0 = 2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}\xi_0\, e^{-ix_e^b \bar{R}\kappa_o R x_e^c - i\frac{4}{\theta'^2} p_e^b \kappa_e^{-1} p_e^c}\,. \tag{2.134}$$

Then the apparent divergence (2.103) of the coefficient at the limit $\kappa_e = e, \kappa_o = o, N = \infty$ does not occur, since

$$|(\bar{R}\kappa_o R)_{ee'}| = \left| \frac{16\, i^{e+e'}\, (ee')^2}{\pi^2} \sum_{o=1}^{\infty} \frac{1}{o(e^2-o^2)(e'^2-o^2)} \right| \quad < \infty\,. \tag{2.135}$$

Following the ideas of the previous subsection, the differential operator L_0 can be represented in terms of star products by introducing the field $\mathcal{L}_0$ and the "midpoint correction" γ as follows

$$\begin{aligned} L_0 \hat{A} &= \mathcal{L}_0 \star \hat{A} + \hat{A} \star \mathcal{L}_0 + \gamma \hat{A}\,, && (2.136)\\ \mathcal{L}_0 &= i \sum_{e>0}\left(\frac{\kappa_e^2}{2} x_e^b x_e^c + \frac{2}{\theta'^2} p_e^b p_e^c\right) + \frac{1}{2}\left(\sum_{e>0}\kappa_e + \sum_{o>0}\kappa_o\right) + \frac{i}{\theta'}\left(\sum_{e>0} w_e p_e^b\right)\xi_0 && (2.137)\\ &= \sum_{e>0}\kappa_e\left(\beta_{-e}^b \star \beta_e^c + \beta_{-e}^c \star \beta_e^b\right) - \frac{1}{2}\left(\sum_{e>0}\kappa_e - \sum_{o>0}\kappa_o\right) + \frac{i}{\theta'}\left(\sum_{e>0} w_e p_e^b\right)\xi_0\,, && (2.138)\\ \gamma &= -\frac{i}{1+\bar{w}w}\left(\sum_{e>0} w_e \frac{\partial}{\partial x_e^b}\right)\left(\sum_{e'>0} w_{e'}\frac{\partial}{\partial x_{e'}^c}\right) && (2.139) \end{aligned}$$

where $\beta_e^{b,c}$ can be rewritten in terms of the even mode variables in eq. (2.132)

$$\beta_e^b = \frac{1}{\theta'} p_{|e|}^c - \frac{i}{2}\epsilon(e)\kappa_e x_{|e|}^b\,, \qquad \beta_e^c = \frac{1}{2} x_{|e|}^c - \frac{i}{\theta'}\epsilon(e)\kappa_e^{-1} p_{|e|}^b\,. \tag{2.140}$$

[10] We used these even modes in [12]. They are just a linear transformation of the odd modes in section 2.3 and different from the even modes which are considered in appendix D.

The ξ_0-dependent term vanishes when L_0 acts on the fields in the Siegel gauge. The field $\mathcal{L}_0$ is multiplied with the star product, but γ is still a differential operator. It is possible to rewrite γ in terms of star products (a double supercommutator)

$$\gamma\,\hat{A} = -\frac{i}{\theta'^2(1+\bar{w}w)} \sum_{e,e'>0} w_e w_{e'} [p_e^b, [p_{e'}^c, \hat{A}\}_\star\}_\star\,, \tag{2.141}$$

but this does not have the single star product structure as the star products with $\mathcal{L}_0$ and therefore γ cannot be absorbed into a redefinition of $\mathcal{L}_0$. As discussed in [12], γ depends only on a single combination of modes in the direction of the vector w_e, which is closely related to the midpoint. If it were not for the "midpoint correction" term γ, string field theory would reduce to a matrix-like theory that would be exactly solvable, as shown in [12]. The above form of writing L_0 focuses on the γ term as the remaining difficult aspect of string field theory.

A similar structure exists in the canonically equivalent odd basis by using $\mathcal{L}_0', \gamma'$

$$\begin{aligned}
L_0\hat{A} &= \mathcal{L}_0' \star \hat{A} + \hat{A} \star \mathcal{L}_0' + \gamma' \hat{A}\,,\\
\mathcal{L}_0' &= i\sum_{o>0} \kappa_o \left(\frac{1}{2} x_o y_o + \frac{2}{\theta'^2} p_o q_o\right) + \frac{1}{2}\left(\sum_{e>0}\kappa_e + \sum_{o>0}\kappa_o\right) + (1+\bar{w}w)\frac{i}{\theta'}\left(\sum_{o>0} v_o \kappa_o p_o\right)\xi_0\\
&= \sum_{o>0}\kappa_o\left(\beta_{-o}^b \star \beta_o^c + \beta_{-o}^c \star \beta_o^b\right) + \frac{1}{2}\left(\sum_{e>0}\kappa_e - \sum_{o>0}\kappa_o\right) + (1+\bar{w}w)\frac{i}{\theta'}\left(\sum_{o>0} v_o\kappa_o p_o\right)\xi_0\,,\\
\gamma' &= \frac{4i}{\theta'^2}(1+\bar{w}w)\left(\sum_{o>0}\kappa_o v_o p_o\right)\left(\sum_{o'>0} v_{o'} q_{o'}\right)\,.
\end{aligned} \tag{2.142}$$

3. Monoid and Neumann coefficients

The perturbative states and nonperturbative squeezed states in Fock space (perturbative vacuum, sliver state, butterfly state and so on) are mapped to gaussian functions in Moyal space [9]. In fermionic Moyal space, such as the basis $\xi = \left(x_e^b, p_e^b, x_e^c, p_e^c\right)$, the generic form of such a gaussian is written as,

$$A_{\mathcal{N},M,\lambda}(\xi) = \mathcal{N} e^{-\bar{\xi}M\xi - \bar{\xi}\lambda}\,, \qquad \bar{M} = -M\,. \tag{3.1}$$

As in the bosonic case, such shifted gaussians form a monoid algebra [9].[11] A monoid is almost a group except for the property of inverse. This implies that under star products the shifted gaussians satisfy the properties of closure, identity and associativity. Although the generic gaussian has also an inverse under star products, not all of them do (for example, projectors such as eq. (2.127) do not have an inverse). The identity element under star products is the natural number 1, which corresponds to the trivial gaussian.

The monoid structure is an effective tool for computations in MSFT. In particular it was used to calculate the product of n gaussian functions, whose trace gives the n-point

[11] Some issues on Moyal product are discussed in [20].

vertex. A corollary of this result is the determination of the Neumann coefficients for the n-string vertex. These coefficients were computed in [2] from conformal field theory, but now they can be determined by using only the Moyal product. The MSFT approach gives simple expressions for all Neumann coefficients in terms of a single matrix $t_{eo} = \kappa_e^{1/2} T_{eo} \kappa_o^{-1/2}$. Neumann coefficients are not needed for computations in MSFT, but the computation can be used to test the MSFT formalism. This was used as successful test of MSFT in the matter sector [9].

In this section we will carry out a similar program in the ghost sector. While the treatment of the unity elements becomes more subtle because of the zero mode, the closure of gaussian functions (3.1) under the star product will be proved exactly the same way as in the bosonic sector. In particular, the algebraic structure is formally the same. With this information, one can compute the product of n fermionic gaussians and derive the Neumann coefficients by using this algebraic machinery.

In particular, we verify consistency of the Moyal $\star$ product (2.95) with the conventional reflector and the 3-string vertex in oscillator language. We will show the correspondence by including the zero mode part,

$$\langle \Psi_1 | \Psi_2 \rangle \leftrightarrow \int d\xi_0 \, \mathrm{Tr} \left(\hat{A}_1(\xi_0, \xi) \star \hat{A}_2(\xi_0, \xi) \right) , \tag{3.2}$$

$$\langle \Psi_1 | \Psi_2 \star^W \Psi_3 \rangle \leftrightarrow \int d\xi_0^{(3)} d\xi_0^{(2)} d\xi_0^{(1)} \, \mathrm{Tr} \left(\hat{A}_1(\xi_0^{(1)}, \xi) \star \hat{A}_2(\xi_0^{(2)}, \xi) \star \hat{A}_3(\xi_0^{(3)}, \xi) \right) . \tag{3.3}$$

3.1 Monoid structure for gaussian elements

We first derive the structure of the monoid (3.1) under the fermionic star product defined by $\{\xi_a, \xi_b\} = \Sigma_{ab}$ for a general symmetric Σ_{ab}. This matrix takes a block diagonal form $\Sigma = \mathrm{diag}\,(\sigma', \sigma')$ in the bases we discussed so far, but it is useful to develop the formalism for any Σ. For the moment, we suppress the ξ_0 -dependence because it is not relevant to the Moyal $\star$-product (2.95). The structure of the monoid is summarized in the following algebra,

$$\begin{aligned}
A_{\mathcal{N}_1, M_1, \lambda_1} \star A_{\mathcal{N}_2, M_2, \lambda_2} &= A_{\mathcal{N}_{12}, M_{12}, \lambda_{12}} \,, && (3.4)\\
m_i := M_i \Sigma \,, \qquad \bar{m} &= -\Sigma m \Sigma^{-1} \,, && (3.5)\\
m_{12} = M_{12}\Sigma &= (1+m_2) m_1 (1+m_2 m_1)^{-1} + (1-m_1) m_2 (1+m_1 m_2)^{-1} \,, && (3.6)\\
\lambda_{12} &= (1-m_1)(1+m_2 m_1)^{-1} \lambda_2 + (1+m_2)(1+m_1 m_2)^{-1} \lambda_1 \,, && (3.7)\\
\mathcal{N}_{12} &= \mathcal{N}_1 \mathcal{N}_2 \det{}^{\frac{1}{2}} (1+m_2 m_1)\, e^{-\frac{1}{4}\sum_{a,b=1}^{2} \bar{\lambda}_a \Sigma K_{ab} \lambda_b} \,, && (3.8)\\
K_{ab} &= \begin{pmatrix} (m_2^{-1}+m_1)^{-1} & (1+m_2 m_1)^{-1} \\ -(1+m_1 m_2)^{-1} & (m_2+m_1^{-1})^{-1} \end{pmatrix} . && (3.9)
\end{aligned}$$

To prove this formula, it is convenient to use Fourier transformation. We define

$$\tilde{A}(\eta) := \int d\xi \, e^{\bar{\xi}\eta} A(\xi) \,, \qquad A(\xi) = \int d\eta \, e^{-\bar{\xi}\eta} \tilde{A}(\eta) \,. \tag{3.10}$$

The $\star$ product is rewritten in terms of Fourier coefficients,

$$A_1 \star A_2(\xi) = \int d\eta_1 d\eta_2 e^{-\frac{1}{2}\bar{\eta}_1 \Sigma \eta_2 - \bar{\xi}\eta_1 - \bar{\xi}\eta_2} \tilde{A}_1(\eta_1) \tilde{A}_2(\eta_2) \,. \tag{3.11}$$

For the gaussian $A_{\mathcal{N},M,\lambda}$ (3.1), the Fourier transform is also gaussian:

$$\widetilde{A_{\mathcal{N},M,\lambda}}(\eta) = \mathcal{N}(\det(2M))^{\frac{1}{2}}\, e^{-\frac{1}{4}\bar{\lambda}M^{-1}\lambda} e^{-\frac{1}{4}\bar{\eta}M^{-1}\eta+\frac{1}{2}\bar{\eta}M^{-1}\lambda}\,. \tag{3.12}$$

The main result (3.4)–(3.9) follows by carrying out the gaussian integration over fermionic variables.

We note that eq. (3.1) have *exactly* the same form as the bosonic case (eqs.(3.11)–(3.17) in [9]) if we put $d = -2$. Similarly, the formula for the trace also related to the bosonic case as if $d = -2$

$$\mathrm{Tr}(A(\xi)) = \mathcal{N}(\det(2m))^{\frac{1}{2}} e^{-\frac{1}{4}\bar{\lambda}M^{-1}\lambda}\,. \tag{3.13}$$

With this observation, we find that all the algebraic manipulations in [9] which use the structure of the star product for monoids also apply in the ghost sector with no other modification.

In particular the product of n monoids with the same M, but different $\lambda_i, \mathcal{N}_i$ is one of the useful results that is used to compute the n-point vertex. The result in [9] is now adopted to the ghost sector as follows

$$A_{\mathcal{N}_{12\cdots n},M^{(n)},\lambda_{12\cdots n}} := A_{\mathcal{N}_1,M,\lambda_1} \star A_{\mathcal{N}_2,M,\lambda_2} \star \cdots \star A_{\mathcal{N}_n,M,\lambda_n}\,, \tag{3.14}$$

$$m^{(n)} = M^{(n)}\Sigma = \frac{J_n^-}{J_n^+}\,, \qquad J_n^{\pm} := \frac{(1+m)^n \pm (1-m)^n}{2}\,, \tag{3.15}$$

$$\lambda_{12\cdots n} = \frac{1}{J_n^+}\sum_{r=1}^{n}(1-m)^{r-1}(1+m)^{n-r}\lambda_r\,, \tag{3.16}$$

$$\mathcal{N}_{12\cdots n} = \mathcal{N}_1\mathcal{N}_2\cdots\mathcal{N}_n(\det J_n^+)^{\frac{1}{2}} \exp\left(-\frac{1}{4}K_n(\lambda)\right)\,, \tag{3.17}$$

$$\begin{aligned} K_n(\lambda) &= \sum_{r=1}^{n}\bar{\lambda}_r\Sigma\frac{J_{n-1}^-}{J_n^+}\lambda_r + \\ &\quad +2\sum_{r<s}^{n}\bar{\lambda}_r\Sigma\frac{(1-m)^{s-r-1}(1+m)^{n+r-s-1}}{J_n^+}\lambda_s\,, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \mathrm{Tr}(A_{\mathcal{N}_{12\cdots n},M^{(n)},\lambda_{12\cdots n}}) &= \mathcal{N}_1\mathcal{N}_2\ldots\mathcal{N}_n \det^{\frac{1}{2}}(2J_n^-) \times \\ &\quad \times \exp\left(-\frac{1}{4}\sum_{r,s=1}^{n}\bar{\lambda}_r\Sigma\mathcal{O}^{(n)}_{(s-r)}(m)\lambda_s\right)\,, \end{aligned} \tag{3.19}$$

$$\mathcal{O}^{(n)}_{(r)}(m) := \mathcal{O}^{(n)}_{r \bmod n}\,, \tag{3.20}$$

$$\mathcal{O}^{(n)}_0(m) = \frac{J_{n-1}^+}{J_n^-} = \frac{(1+m)^{n-1}+(1-m)^{n-1}}{(1+m)^n-(1-m)^n}\,, \tag{3.21}$$

$$\begin{aligned} \mathcal{O}^{(n)}_i(m) &= \frac{(1+m)^{n-i-1}(1-m)^{i-1}}{J_n^-} \\ &= \frac{2(1+m)^{n-i-1}(1-m)^{i-1}}{(1+m)^n-(1-m)^n}\,, \quad (1 \le i \le n-1)\,. \end{aligned} \tag{3.22}$$

This will be used in the next section to compute the Neumann coefficients as a by-product.

JHEP07(2003)027

The algebraic structure is also used to construct projectors (such as sliver state or butterfly). As in [9], the generic form of the projector in the ghost sector can be written as a particular class of gaussian functions

$$A_{D,\lambda}(\xi) = 2^{-2N}\, e^{\frac{1}{4}\bar{\lambda}\Sigma D\Sigma\lambda} e^{-\bar{\xi}D\xi-\bar{\xi}\lambda}\,, \qquad (D\Sigma)^2 = 1\,, \tag{3.23}$$

$$A_{D,\lambda} \star A_{D,\lambda}(\xi) = A_{D,\lambda}(\xi)\,, \qquad \mathrm{Tr}(A_{D,\lambda}(\xi)) = 1\,. \tag{3.24}$$

3.2 Neumann coefficients

In this section, we construct the Neumann coefficients by using the MSFT formalism for the n-point vertices given above. The basic idea is to use the correspondence of vertices in the operator formalism and in MSFT given by [9],

$$_1\langle\Psi_1| \otimes \cdots \otimes {}_n\langle\Psi_n|V_n\rangle \sim \mathrm{Tr}(\hat{A}_1 \star \cdots \star \hat{A}_n) \tag{3.25}$$

where $\hat{A}_r(\xi_0,\xi) = \langle\xi_0,\xi|\tilde{\Psi}_r\rangle$ and $|\tilde{\Psi}_r\rangle = \langle\Psi_r|V_2\rangle$. For the ghost sector, we have to be careful in the treatment of the zero mode in (3.25). We use this identification to express Neumann coefficients by taking the following steps.

1. In Fock space we choose n coherent states for $\langle\Psi_r|$, $r = 1, 2, \ldots n$, labelled by parameters $\mu^{*(r)}$. The left hand side of (3.25) can be computed in the operator formalism. The result takes the form of an exponential that contains a quadratic form in the parameters of the coherent states $\mu^{(r)}$. The Neumann coefficients that define $|V_n\rangle$ appear as the coefficients in the quadratic form. We treat the Neumann coefficients as unknown matrices.

2. We calculate $\hat{A}_r(\xi_0,\xi) = \langle\xi_0,\xi|\tilde{\Psi}_r\rangle$ which gives the Moyal image of the coherent states in the form of monoids, with the $\lambda^{(r)}$ related to the parameters $\mu^{*(r)}$ of the coherent state.

3. We compute the right hand side of (3.25) by using the result in eqs. (3.14)–(3.22). As in item (1) this is also an exponential containing a quadratic form in the coherent state parameters $\mu^{*(r)}$, but with the coefficients determined by the monoid algebra given above.

4. We compare the coefficients of the parameters in both sides and thus determine the Neumann coefficients completely. They turn out to be simple functions of a single matrix $t_{eo} = \kappa_e^{1/2} T_{eo} \kappa_o^{-1/2}$.

Throughout this subsection, we use the regularized framework (N, κ_e, κ_o) to make the algebraic manipulation consistent. This gives a new generalization of Neumann coefficients since the new expression includes arbitrary spectral parameters and arbitrary N. To compare to the Neumann coefficients computed through conformal field theory, the open string limit $(N = \infty, \kappa_e = e, \kappa_o = o)$ is taken at the end. Through analytic and numerical methods it is shown that these very different looking forms of Neumann coefficients are indeed the same. This successful test of MSFT provides confidence about its correctness and shows that MSFT is an alternative tool for computation in string theory.

JHEP07(2003)027

Coherent states. Coherent states $\langle\Psi|$ are defined by

$$\langle\Psi|\hat{b}^\dagger = \langle\Psi|\mu_b^*\,, \qquad \langle\Psi|\hat{c}^\dagger = \langle\Psi|\mu_c^*\,, \qquad \langle\Psi|\hat{b}_0 = \langle\Psi|\mu_0^*\,. \tag{3.26}$$

They have the following explicit form

$$\langle\Psi| = \langle\Omega|\hat{c}_{-1}e^{\mu_b^*\hat{c}+\mu_c^*\hat{b}+\mu_0^*\hat{c}_0}\,. \tag{3.27}$$

The inner product between the standard n-string vertices $|V_n\rangle$ (appendix C) and the coherent states $\langle\Psi_r|$ is given as follows for $n=1,2,3$

$$n=1:\langle\Psi|I\rangle = \frac{\pi}{2\sqrt{2}}\bar{v}_o\kappa_o\mu_{bo}^*\left(\mu_0^*-\sqrt{2}\bar{w}\mu_{be}^*\right)e^{\bar{\mu}_c^*C\mu_b^*}\,, \tag{3.28}$$

$$\begin{aligned}
n=2:{}_1\langle\Psi_1|_2\langle\Psi_2|V_2\rangle_{12} &= \left(\mu_0^{*(1)}-\mu_0^{*(2)}\right)e^{\bar{\mu}_c^{*(1)}C\mu_b^{*(2)}+\bar{\mu}_c^{*(2)}C\mu_b^{*(1)}}\\
&= \left(\mu_0^{*(1)}-\mu_0^{*(2)}\right)e^{\frac{1}{2}\bar{\mu}^{*(1)}\varepsilon C\mu^{*(2)}+\frac{1}{2}\bar{\mu}^{*(2)}\varepsilon C\mu^{*(1)}}\,,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
n=3:{}_1\langle\Psi_1|_2\langle\Psi_2|_3\langle\Psi_3|V_3\rangle_{123} &= \exp\left(-\bar{\mu}_c^{*(r)}X^{rs}\mu_b^{*(s)}-\bar{\mu}_c^{*(r)}X^{rs}{}_0\mu_0^{*(s)}\right)\\
&= \exp\left(-\frac{1}{2}\bar{\mu}^{*(r)}\varepsilon\mathcal{X}^{rs}\mu^{*(s)}-\bar{\mu}_c^{*(r)}X^{rs}{}_0\mu_0^{*(s)}\right)\,,
\end{aligned} \tag{3.30}$$

where

$$\mu^* = \begin{pmatrix}\mu_c^*\\ \mu_b^*\end{pmatrix},\quad \varepsilon = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \mathcal{X}^{rs} = \begin{pmatrix}\bar{X}^{sr} & 0\\ 0 & X^{rs}\end{pmatrix},\quad C = \begin{pmatrix}C & 0\\ 0 & C\end{pmatrix},\quad C_{nm} = (-1)^n\delta_{n,m}\,. \tag{3.31}$$

As we noted, the Neumann coefficients appear as the coefficients of quadratic functions of $\mu^{*(r)}$ in the exponential.

Moyal image of coherent state. We define a Moyal field which corresponds to the coherent state $\langle\Psi|$. First, we have the corresponding ket $|\tilde{\Psi}\rangle$ by using the reflector (C.4):

$$|\tilde{\Psi}\rangle_1 := {}_2\langle\Psi|V_2\rangle_{12} = e^{\hat{c}_0^{(1)}\mu_0^*+\hat{c}^{\dagger(1)}C\mu_b^*+\mu_c^*C\hat{b}^{\dagger(1)}}\hat{c}_1^{(1)}|\Omega\rangle_1\,. \tag{3.32}$$

Then we get the Moyal field by using the ket $\langle\xi_0,\xi|$ (2.99) which defines the Moyal basis:

$$\hat{A}(\xi_0,\xi) := \langle\xi_0,\xi|\tilde{\Psi}\rangle = -2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}(\xi_0e^{-\sqrt{2}\bar{\mu}_c^*w\mu_0^*}+\mu_0^*)e^{-\frac{1}{2}\bar{\mu}^*\varepsilon C\mu^*-\bar{\xi}M_0\xi-\bar{\xi}\lambda} \tag{3.33}$$

where we denoted

$$M_0 = \begin{pmatrix}0 & iM_0^{(o)}\\ -i\bar{M}_0^{(o)} & 0\end{pmatrix},\qquad \lambda = \begin{pmatrix}-i\sqrt{2}\mu_{co}^*\\ \frac{-2\sqrt{2}i}{\theta'}\bar{S}\mu_{be}^*+\frac{2i}{\theta'}\bar{S}w\xi_0\\ \sqrt{2}\mu_{bo}^*\\ \frac{2\sqrt{2}}{\theta'}R\mu_{ce}^*\end{pmatrix} = 2K^*(\mu^*+W\xi_0)\,, \tag{3.34}$$

$$K^* = \begin{pmatrix}0 & -\frac{i}{\sqrt{2}} & 0 & 0\\ 0 & 0 & -\frac{\sqrt{2}i}{\theta'}\bar{S} & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}\\ \frac{\sqrt{2}}{\theta'}R & 0 & 0 & 0\end{pmatrix},\qquad W = \begin{pmatrix}0\\ 0\\ -\frac{1}{\sqrt{2}}w\\ 0\end{pmatrix}. \tag{3.35}$$

It takes the form of the standard element of monoid although with the pre-factor $\mathcal{N}$ and the λ in the exponent depend on the zero mode ξ_0. We can apply the Moyal $\star$ product formula for monoids which was developed in the previous subsection.

Explicit form of n-th product. After using the results of (3.14)–(3.22), we have obtained the trace formula for n-th product of Moyal fields which correspond to coherent states

$$\begin{aligned}
&\mathrm{Tr}(\hat{A}_1(\xi_0^{(1)},\xi)\star\hat{A}_2(\xi_0^{(2)},\xi)\star\cdots\star\hat{A}_n(\xi_0^{(n)},\xi))=\\
&\quad=(-1)^n 2^{-2nN}(1+\bar{w}w)^{-\frac{n}{4}}\det{}^{\frac{1}{2}}(2J_n^-)\prod_{r=1}^{n}\left(\xi_0^{(r)}e^{-\sqrt{2}\bar{\mu}_c^{*(r)}w\mu_0^{*(r)}}+\mu_0^{*(r)}\right)\times\\
&\qquad\times e^{-\frac{1}{2}\sum_{r=1}^{n}\bar{\mu}^{*(r)}\varepsilon C\mu^{*(r)}-\frac{1}{4}\sum_{r,s=1}^{n}\bar{\lambda}^{(r)}\Sigma\mathcal{O}_{(s-r)}(m_0)\lambda^{(s)}}\\
&\quad=(-1)^n 2^{-2nN}(1+\bar{w}w)^{-\frac{n}{4}}\det{}^{\frac{1}{2}}(2J_n^-)\prod_{r}\left(\xi_0^{(r)}e^{-\sqrt{2}\bar{\mu}_c^{*(r)}w\mu_0^{*(r)}}+\mu_0^{*(r)}\right)\times\\
&\qquad\times\exp\left(\frac{1}{2}\sum_{r,s=1}^{n}\bar{\mu}^{*(r)}\varepsilon C(K^{*-1}2m_0\mathcal{O}_{(s-r)}(m_0)K^*-\delta^{r,s})\mu^{*(s)}\right)\\
&\qquad\times\exp\left(\sum_{r,s=1}^{n}\bar{\mu}^{*(r)}\varepsilon CK^{*-1}2m_0\mathcal{O}_{(s-r)}(m_0)K^*W\xi_0^{(s)}\right),
\end{aligned}\tag{3.36}$$

where we used the relations[12]

$$\bar{K}^*\Sigma=-\varepsilon CK^{*-1}m_0\,,\qquad m_0:=M_0\Sigma\,,\tag{3.37}$$

$$K^{*-1}m_0K^*=\begin{pmatrix}\bar{\hat{m}}_0^{*-1} & 0\\ 0 & \hat{m}_0^{*-1}\end{pmatrix},\qquad \hat{m}_0^*:=\sqrt{\kappa}\tilde{m}_0^*\frac{1}{\sqrt{\kappa}}=\begin{pmatrix}0 & -S\\ -\bar{T} & 0\end{pmatrix},\tag{3.38}$$

$$\bar{W}\varepsilon CK^{*-1}2m_0\mathcal{O}_{(s-r)}(m_0)K^*W=0\,.\tag{3.39}$$

Here $\tilde{m}_0^*$ which defines $\hat{m}_0^*$ has appeared in eq. (A.62) in the context of the matter sector in MSFT. We should emphasize that although we are using a notation similar to the one in the matter sector, the meaning of m_0 here is *not* the same as the one in the matter sector which is defined by eq. (A.60). However, because there are some relationships between the m_0 in the matter and ghost sectors there are some relations among Neumann coefficients in these sectors. Note that in eq. (3.36) we assigned different ξ_0's for each Moyal field in the trace. This prescription is necessary to find agreement with Witten's star product as we will see soon. Noting

$$\xi_0^{(r)}e^{-\sqrt{2}\bar{\mu}_c^{*(r)}w\mu_0^{*(r)}}+\mu_0^{*(r)}=\delta(\xi_0^{(r)}+\mu_0^{*(r)})\,e^{-\sqrt{2}\bar{\mu}_c^{*(r)}w\mu_0^{*(r)}}=\delta(\xi_0^{(r)}+\mu_0^{*(r)})\,e^{2\bar{\mu}^{*(r)}\varepsilon CW\mu_0^{*(r)}}\,,\tag{3.40}$$

we perform the $\xi_0^{(r)}$-integrations for all zero modes, and find

$$\begin{aligned}
&\int d\xi_0^{(n)}d\xi_0^{(n-1)}\cdots d\xi_0^{(1)}\,\mathrm{Tr}(\hat{A}_1(\xi_0^{(1)},\xi)\star\hat{A}_2(\xi_0^{(2)},\xi)\star\cdots\star\hat{A}_n(\xi_0^{(n)},\xi))\\
&\quad=(-1)^n 2^{-2nN}(1+\bar{w}w)^{-\frac{n}{4}}\det{}^{\frac{1}{2}}(2J_n^-)\times\\
&\qquad\times\exp\left(\frac{1}{2}\sum_{r,s=1}^{n}\bar{\mu}^{*(r)}\varepsilon C(K^{*-1}2m_0\mathcal{O}_{(s-r)}(m_0)K^*-\delta^{r,s})\mu^{*(s)}\right)\times
\end{aligned}$$

[12]Similar relations were used to obtain the Neumann coefficients in matter sector. (appendix A.2.4)

$$\times \exp\left(\sum_{r,s=1}^{n}\bar{\mu}^{*(r)}\varepsilon C(2\delta^{r,s}-K^{*-1}2m_0\mathcal{O}_{(s-r)}(m_0)K^*)W\mu_0^{*(s)}\right). \tag{3.41}$$

Comparison of coefficients. Now, we consider the above formulas in the cases $n = 1, 2, 3$.

$n = 1$ **case:** eq. (3.41) becomes

$$\int d\xi_0 \,\mathrm{Tr}\,\hat{A}(\xi_0,\xi) = -(1+\bar{w}w)^{\frac{1}{4}} e^{\mu_c^* C\mu_b^*}\,. \tag{3.42}$$

In this case from eq. (3.28) there is a correspondence

$$\int d\xi_0 \,\mathrm{Tr}\,\hat{A}(\xi_0,\xi) \sim \langle\Psi|I\rangle\,, \tag{3.43}$$

up to pre-factor which comes from the b-ghost insertion in conventional operator formalism. [2] On the other hand, up to a constant factor, we have

$$\int d\xi_0\delta(\xi_0)\,\mathrm{Tr}\,\hat{A}(\xi_0,\xi) \sim \langle\Psi|\tilde{I}\rangle = \mu_0^*\, e^{\mu_c^* C\mu_b^*}\,, \tag{3.44}$$

for the identify-like state $|\tilde{I}\rangle$ which corresponds to the *identity* element for the reduced product [21, 22]. In MSFT, the Moyal field $\langle\xi_0,\xi|\tilde{I}\rangle$ is the identity element in the Siegel gauge. In this sense, $|\tilde{I}\rangle$ appears naturally rather than the BRST-invariant $|I\rangle$ in the context of MSFT.

$n = 2$ **case:** Using eq. (3.36), we get

$$\begin{aligned}&\int d\xi_0^{(2)}d\xi_0^{(1)}\delta\left(\xi_0^{(1)}-\xi_0^{(2)}\right)\times\\ &\times\mathrm{Tr}\left(\hat{A}_1\left(\xi_0^{(1)},\xi\right)\star\hat{A}_2\left(\xi_0^{(2)},\xi\right)\right)=\left(\mu_0^{*(1)}-\mu_0^{*(2)}\right)e^{\frac{1}{2}\bar{\mu}^{*(1)}\varepsilon C\mu^{*(2)}+\frac{1}{2}\bar{\mu}^{*(2)}\varepsilon C\mu^{*(1)}}.\end{aligned} \tag{3.45}$$

From eq. (3.29) we have obtained the correspondence

$$\int d\xi_0^{(2)}d\xi_0^{(1)}\delta\left(\xi_0^{(1)}-\xi_0^{(2)}\right)\mathrm{Tr}\left(\hat{A}_1\left(\xi_0^{(1)},\xi\right)\star\hat{A}_2\left(\xi_0^{(2)},\xi\right)\right) = {}_1\langle\Psi_1|_2\langle\Psi_2|V_2\rangle_{12}\,. \tag{3.46}$$

In this case the normalization also coincides. We can interpret $\int d\xi_0^{(2)}d\xi_0^{(1)}\delta(\xi_0^{(1)}-\xi_0^{(2)})$ as the pre-factor $(c_0^{(1)}+c_0^{(2)})$ in the form of ${}_1\langle c_0^{(1)},x_n^{(1)},y_n^{(1)}|_2\langle c_0^{(2)},x_n^{(2)},y_n^{(2)}|V_2\rangle_{12}$ (C.5). Eq. (3.46) can be rewritten as

$$\int d\xi_0\mathrm{Tr}(\hat{A}_1(\xi_0,\xi)\star\hat{A}_2(\xi_0,\xi)) = \langle\Psi_1|\tilde{\Psi}_2\rangle\,, \tag{3.47}$$

for $|\tilde{\Psi}_2\rangle_1 := {}_2\langle\Psi_2|V_2\rangle_{12}$. This is consistent with the normalization (2.98) which we adopted to fix the map from the conventional field to the Moyal field (2.94).

$n = 3$ case: We can identify the Neumann coefficients for the nonzero modes by comparing

$$\mathrm{Tr}\left(\hat{A}_1\left(\xi_0^{(1)},\xi\right)\star\hat{A}_2\left(\xi_0^{(2)},\xi\right)\star\hat{A}_3\left(\xi_0^{(3)},\xi\right)\right)\sim\langle\Psi_1|\langle\Psi_2|\langle\Psi_3|V_3\rangle\,. \tag{3.48}$$

From eqs. (3.36), (3.30) we get the Neumann coefficients in MSFT:

$$\mathcal{X}^{rs}=-C\left(K^{*-1}2m_0\mathcal{O}_{(s-r)}(m_0)K^*-\delta^{r,s}\right).$$

More explicitly

$$X^{(0)}=-C\frac{\hat{m}_0^{*-2}-1}{\hat{m}_0^{*-2}+3},\qquad X^{(+)}=-C\frac{2(1+\hat{m}_0^{*-1})}{\hat{m}_0^{*-2}+3},\qquad X^{(-)}=-C\frac{2(1-\hat{m}_0^{*-1})}{\hat{m}_0^{*-2}+3}\,. \tag{3.49}$$

To identify the Neumann coefficients including the zero mode part, we should perform the ξ_0 integration: $\int d\xi_0^{(3)}d\xi_0^{(2)}d\xi_0^{(1)}$. We can interpret that this comes from the pre-factor:

$$\left(c_0^{(1)}-\bar{w}y_e^{(1)}\right)\left(c_0^{(2)}-\bar{w}y_e^{(2)}\right)\left(c_0^{(3)}-\bar{w}y_e^{(3)}\right)=\bar{c}^{(1)}\bar{c}^{(2)}\bar{c}^{(3)}$$

in the form

$${}_1\langle c_0^{(1)},x_n^{(1)},y_n^{(1)}|_2\langle c_0^{(2)},x_n^{(2)},y_n^{(2)}|_1\langle c_0^{(3)},x_n^{(3)},y_n^{(3)}|V_2\rangle_{123}$$

(C.7). In fact, by identifying[13] eq. (3.41) with eq. (3.30):

$$\int d\xi_0^{(3)}d\xi_0^{(2)}d\xi_0^{(1)}\mathrm{Tr}\left(\hat{A}_1\left(\xi_0^{(1)},\xi\right)\star\hat{A}_2\left(\xi_0^{(2)},\xi\right)\star\hat{A}_3\left(\xi_0^{(3)},\xi\right)\right)\sim\langle\Psi_1|\langle\Psi_2|\langle\Psi_3|V_3\rangle \tag{3.50}$$

up to constant factor, we obtain[14]

$$\begin{aligned}
X^{(0)}_0&=-2\varepsilon CW+\varepsilon CK^{*-1}\frac{2+2m_0^2}{m_0^2+3}K^*W=\frac{4}{\hat{m}_0^{*-2}+3}\frac{w}{\sqrt{2}},\\
X^{(+)}_0&=\varepsilon CK^{*-1}\frac{2+2m_0}{m_0^2+3}K^*W=-\frac{2-2\hat{m}_0^{*-1}}{\hat{m}_0^{*-2}+3}\frac{w}{\sqrt{2}},\\
X^{(-)}_0&=\varepsilon CK^{*-1}\frac{2-2m_0}{m_0^2+3}K^*W=-\frac{2+2\hat{m}_0^{*-1}}{\hat{m}_0^{*-2}+3}\frac{w}{\sqrt{2}}\,.
\end{aligned} \tag{3.51}$$

The Neumann coefficients $X^{(0,\pm)}, X^{(0,\pm)}_0$ agree with eq.(A.68) which was obtained by using the trace of 6 coherent states in the *matter* sector. This implies that the Gross-Jevicki nonlinear relations for Neumann coefficients in MSFT are all satisfied for arbitrary (N,κ_e,κ_o), as was shown in [9]. Namely, our Moyal star product is consistent with the conventional Witten star product in both the matter and ghost sectors.[15] We have confirmed the correspondence between Moyal $\star$ product in MSFT and Witten's one ($\star^W$):

$$\int d\xi_0^{(3)}d\xi_0^{(2)}d\xi_0^{(1)}\mathrm{Tr}\left(\hat{A}_1\left(\xi_0^{(1)},\xi\right)\star\hat{A}_2\left(\xi_0^{(2)},\xi\right)\star\hat{A}_3\left(\xi_0^{(3)},\xi\right)\right)\quad\leftrightarrow\quad\langle\Psi_1|\Psi_2\star^W\Psi_3\rangle \tag{3.52}$$

[13] The ghost zeromode ξ_0 dependence is similar to momentum p_0 dependence in matter sector. But in this case, we do not need "momentum conservation factor" $\delta(\xi_0^{(1)}+\xi_0^{(2)}+\xi_0^{(3)})$. This fact correspond to the lack of $\delta(b_0^{(1)}+b_0^{(2)}+b_0^{(3)})$ factor in ref. [24] eq.(2.18) which gives the correct 3-string vertex in oscillator representation.

[14] We used the notation: $w=\binom{w_e}{0}$.

[15] This also implies that Moyal $\star$ product (2.95) is essentially the same as the *reduced* product in [16, 21, 22] which was defined by omitting ghost zero mode-dependence in original Witten's star product.

up to constant factor for $\kappa_e = e, \kappa_o = o, N = \infty$. As we will show in the next subsection we have numerical confirmation that the generalized Neumann coefficients in MSFT for arbitrary κ_e, κ_o, N converge to the conventional one in the operator formalism when we take the limit.

3.3 Numerical comparison of Neumann coefficients

In this subsection we compare the generalized Neumann coefficients derived algebraically in the Moyal star formalism for any κ_e, κ_o, N, with the independent computation from the point of view of conformal field theory [2] valid at $\kappa_e = e, \kappa_o = o, N = \infty$. We summarize the Neumann coefficients computed from CFT in appendix G.1, together with some differences in the convention.

In [9] we have already given an analytic proof that our algebraic expression of Neumann coefficients coincides with the exact value in [2] in the limit, by comparing the spectroscopy of Neumann coefficients. Namely, in our case by diagonalizing the matrix $t_{eo} = \kappa_e^{1/2} T_{eo} \kappa_o^{-1/2}$ we diagonalize the Neumann coefficients for n-point vertices, since they all depend on the same matrix t. The eigenvalues obtained in this way for the case of the 3-point vertex in the limit $\kappa_e = e, \kappa_o = o, N = \infty$ coincide with the corresponding eigenvalues obtained from Neumann spectroscopy in [13].

A numerical study provides another approach to confirm that the Moyal star and CFT calculations agree in the limit. In the following numerical analysis we show that there is agreement in the limit, and furthermore that there is a clear universal behavior of the approach to the limit as a function of N. In the table 1 given in appendix G.2 we give the MSFT results for the numerical values of the Neumann coefficient $\mathcal{M}_{ee'}^{(0)}(N)$ in the matter sector, and the Neumann coefficient $X_{ee'}^{(0)}(N)$ in the ghost sector for $e, e' = 2, 4, 6, 8$, at different values of the cut-off parameter N. We set the spectral parameters as $\kappa_e = e, \kappa_o = o$. The expression of the Neumann coefficients in the Moyal star computation is given in (5.32)–(5.34) in [9] for the matter sector, and eqs.(3.49), (3.51) in this paper for the ghost sector. In the table 1 we write the ratio with their limiting value, $\mathcal{M}_{ee'}^{(0)}(N)/\mathcal{M}_{ee'}^{(0)}(cft)$ and $X_{ee'}^{(0)}(N)/X_{ee'}^{(0)}(cft)$, where the limiting value is taken as the CFT value given in [2]. The table 1 at different values of N clearly show the convergence

$$\lim_{N\to\infty} \frac{\mathcal{M}_{ee'}^{(0)}(N)}{\mathcal{M}_{ee'}^{(0)}(cft)} = 1\,, \qquad \lim_{N\to\infty} \frac{X_{ee'}^{(0)}(N)}{X_{ee'}^{(0)}(cft)} = 1\,. \tag{3.53}$$

Namely in the open string limit, the Neumann coefficients derived algebraically in MSFT becomes identical with their analytic value computed in CFT.

We note that the convergence of the Neumann coefficients of the ghost sector is much slower than those of matter sector. However log-log plot of $|\mathcal{M}(N)/\mathcal{M}(cft) - 1|$ against N clearly shows that the deviation scales as power of N with a very good accuracy.

As examples, we write the fitting of $(2,2)$ and $(2,4)$ components of above ratios as[16]

$$\frac{\mathcal{M}_{22}^{(0)}(N)}{\mathcal{M}_{22}^{(0)}(cft)} \sim 1 + 1.33 \cdot N^{-1.34}\,, \qquad \frac{\mathcal{M}_{24}^{(0)}(N)}{\mathcal{M}_{24}^{(0)}(cft)} \sim 1 + 2.38 \cdot N^{-1.36}\,, \tag{3.54}$$

[16]These are based on the numerical data for $N = 20, 50, 100, 200, 400$.

$$\frac{X^{(0)}_{22}(N)}{X^{(0)}_{22}(cft)} \sim 1 + 0.834 \cdot N^{-0.669} \,, \qquad \frac{X^{(0)}_{24}(N)}{X^{(0)}_{24}(cft)} \sim 1 + 1.22 \cdot N^{-0.684} \,. \tag{3.55}$$

We have numerically checked that all the Neumann coefficients including the zero mode behave exactly the same way as above,

$$\frac{\mathcal{M}^{(0,\pm)}(N)}{\mathcal{M}^{(0,\pm)}(cft)} - 1 \sim \alpha^{(0,\pm)m}_{nm} N^{-\beta_m} \,, \qquad \frac{X^{(0,\pm)}(N)}{X^{(0,\pm)}(cft)} - 1 \sim \alpha^{(0,\pm)gh}_{nm} N^{-\beta_{gh}} \,, \tag{3.56}$$

where the coefficients α's are order one quantity which depends on the type of the Neumann coefficients. On the other hand, the power β_m and β_{gh} are universal for matter and ghost sector. In the numerical study so far, $\beta_m \sim 1.33$ and $\beta_{gh} \sim 0.67$ for all types of Neumann coefficients. We suspect that there may be an analytic evaluation of the deviations which will prove such a systematic behavior. In any case, eq. (3.56) gives a useful numerical estimate of the deviation at finite N from the $N = \infty$ values.

4. Applications

In this section, we consider the applications of the Moyal star formulation in the ghost sector. We discuss two topics which are essential in the development of MSFT.

The first issue is the derivation of the *regularized* string field theory action in the Siegel gauge including ghosts

$$S = -\int d^d\bar{x}\, \mathrm{Tr}\left(\frac{1}{2\alpha'} A \star (L_0 - 1) A + \frac{g}{3} A \star A \star A\right). \tag{4.1}$$

The regularized version was the starting point of our recent discussions in [10, 12].

The second issue is the derivation of the Feynman rules in the ghost sector. Together with our previous work on Feynman diagrams in the matter sector [10] this provides the complete set of Feynman rules. We show some explicit examples of computations of amplitudes.

4.1 Regularized MSFT action and equation of motion

We start from Witten's string field theory action in the operator formulation

$$S = \frac{1}{2}\langle\Psi|Q_B|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi \star^W \Psi\rangle \,. \tag{4.2}$$

Q_B is BRST operator, which may be written by separating out the b_0, c_0 zero modes

$$Q_B = \hat{c}_0(L_0 - 1) + 2\hat{X}\hat{b}_0 + \tilde{Q} \,, \tag{4.3}$$

with

$$L_0 = L_0^{\mathrm{matter}} + L_0^{\mathrm{ghost}} := \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \sum_{n=1}^{\infty} n\left(\hat{b}_{-n}\hat{c}_n + \hat{c}_{-n}\hat{b}_n\right), \tag{4.4}$$

$$\hat{X} = -\sum_{n=1}^{\infty} n\hat{c}_{-n}\hat{c}_n \,, \tag{4.5}$$

$$\tilde{Q} = \sum_{n\neq 0} \hat{c}_{-n} L_n^{\text{matter}} + \sum_{m,n,m+n\neq 0} \frac{m-n}{2} \hat{c}_m \hat{c}_n \hat{b}_{-m-n}\,, \tag{4.6}$$

$$L_n^{\text{matter}} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{-m}\alpha_{n+m}\,. \tag{4.7}$$

By imposing the Siegel gauge condition $\hat{b}_0|\Psi\rangle = 0$ we obtain the gauge fixed action

$$S = \frac{1}{2}\langle\Psi|\hat{c}_0(L_0-1)|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi \star^W \Psi\rangle\,. \tag{4.8}$$

In the regularized version of MSFT with cut-off parameters (N, κ_e, κ_o), we cannot write a nilpotent Q_B operator, at least technically for the time being, because the conformal symmetry is explicitly broken when the parameters (N, κ_e, κ_o) are not at their limiting values. Of course, any other approach that attempts to work with a finite number of modes (such as level truncation) suffers from the same problem. For complete control, what seems to be desirable is the construction of a finite dimensional Lie algebra that would be a substitute for the Virasoro algebra at finite N, and which would tend to the Virasoro algebra at infinite N. If such an algebra could be constructed, then a regulated version of Q_B at finite N would be straightforward, at least in MSFT.

On the other hand we have seen in numerous cases by now that the regulator is indispensable. With this restriction, we are forced to work with the gauge fixed action eq. (4.8) where the truncation of the oscillators can be made self-consistently. In the open string limit, we recover the original gauge fixed action which is equivalent to the original gauge invariant action eq. (4.2).

In the following we rewrite the action (4.8) in the Moyal language. We use a field $\hat{A}(\bar{x}, \xi_0, \xi) = \xi_0\, A(\bar{x}, \xi)$ in the Siegel gauge which is related to a conventional string field Ψ by Fourier transformation (A.28) (2.94). The kinetic term is rewritten as,

$$\begin{aligned}
\langle\Psi|\hat{c}_0(L_0-1)|\Psi\rangle &= \int(-d\xi_0)\int d^d\bar{x}\,\mathrm{Tr}\left(\hat{A}(\bar{x},\xi_0,\xi)\star\hat{\beta}_{\hat{c}_0}(L_0-1)\hat{A}(\bar{x},\xi_0,\xi)\right)\\
&= \int d^d\bar{x}\,\mathrm{Tr}\left(A(\bar{x},\xi)\star(L_0-1)A(\bar{x},\xi)\right)
\end{aligned} \tag{4.9}$$

where[17] $L_0 = L_0^{\text{matter}} + L_0^{\text{ghost}}$ is given by

$$\begin{aligned}
L_0^{\text{matter}} &= \frac{1}{2}\beta_0^2 - \frac{d}{2}\mathrm{Tr}\,\tilde{\kappa} - \frac{1}{4}\bar{D}_\xi M_0^{-1}\tilde{\kappa}D_\xi + \bar{\xi}\tilde{\kappa}M_0\xi\,,\\
L_0^{\text{ghost}} &= \mathrm{Tr}\,\tilde{\kappa}^{gh} - \frac{1}{2}\frac{\bar{\partial}}{\partial\xi^b}\left(M_0^{gh}\right)^{-1}\tilde{\kappa}^{gh}\frac{\partial}{\partial\xi^c} + 2\bar{\xi}^b\tilde{\kappa}^{gh}M_0^{gh}\xi^c\\
&= \mathrm{Tr}\,\tilde{\kappa}^{gh} - \frac{1}{4}\frac{\bar{\partial}}{\partial\xi^{gh}}\varepsilon\left(M_0^{gh}\right)^{-1}\tilde{\kappa}^{gh}\frac{\partial}{\partial\xi^{gh}} + \bar{\xi}^{gh}\varepsilon\tilde{\kappa}^{gh}M_0^{gh}\xi^{gh}\,,\\
\beta_0 &= -il_s\frac{\partial}{\partial\bar{x}}\,,\quad D_\xi = \begin{pmatrix} \frac{\partial}{\partial x_e} - i\frac{w_e}{l_s}\beta_0 \\ \frac{\partial}{\partial p_e} \end{pmatrix},
\end{aligned}$$

[17] In this section, we use the variable $\bar{\xi}^{gh} = (\bar{\xi}_e^b, \bar{\xi}_e^c) = (x_e^b, p_e^b, x_e^c, p_e^c)$ which was introduced in eq. (2.132) because this makes L_0^{ghost} most similar to the L_0^{matter}.

$$\tilde{\kappa} = \begin{pmatrix} \kappa_e & 0 \\ 0 & T\kappa_o R \end{pmatrix}, \quad M_0 = \begin{pmatrix} \frac{\kappa_e}{2l_s^2} & 0 \\ 0 & \frac{2l_s^2}{\theta^2} T\kappa_o^{-1}\bar{T} \end{pmatrix},$$

$$\tilde{\kappa}^{gh} = \begin{pmatrix} \bar{R}\kappa_o\bar{T} & 0 \\ 0 & \kappa_e \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M_0^{gh} = \begin{pmatrix} \frac{i}{2}\bar{R}\kappa_o R & 0 \\ 0 & \frac{2i}{\theta'^2}\kappa_e^{-1} \end{pmatrix} \tag{4.10}$$

and the Moyal $\star$ product and the trace are

$$\star = \exp\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\xi}\sigma\frac{\overrightarrow{\partial}}{\partial\xi} + \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\xi^{gh}}\Sigma\frac{\overrightarrow{\partial}}{\partial\xi^{gh}}\right), \qquad \mathrm{Tr} = \frac{\det\sigma'}{|\det(2\pi\sigma)|^{d/2}}\int d^{2Nd}\xi \;\; d^{4N}\xi^{gh},$$

$$\sigma = i\theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma' \end{pmatrix}, \qquad \sigma' = \theta'\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.11}$$

On the other hand, the cubic term of the action becomes

$$\begin{aligned}\langle\Psi|\Psi\star^W\Psi\rangle &= \mu_3^{-1}\int d\xi_0^{(3)}d\xi_0^{(2)}d\xi_0^{(1)}\int d^d\bar{x}\,\mathrm{Tr}\left(\hat{A}\left(\bar{x},\xi_0^{(1)},\xi\right)\star\hat{A}\left(\bar{x},\xi_0^{(2)},\xi\right)\star\hat{A}\left(\bar{x},\xi_0^{(3)},\xi\right)\right) \\ &= \mu_3^{-1}\int d^d\bar{x}\,\mathrm{Tr}\left(A\left(\bar{x},\xi\right)\star A\left(\bar{x},\xi\right)\star A\left(\bar{x},\xi\right)\right)\end{aligned} \tag{4.12}$$

where

$$\mu_3 = -2^{2N(d-2)}(1+\bar{w}w)^{-\frac{d}{8}+\frac{3}{4}}(\det(3+t\bar{t}))^{-d}(\det(1+3t\bar{t}))^2, \qquad t := \kappa_e^{1/2}T\kappa_o^{-1/2}. \tag{4.13}$$

After an appropriate rescaling of A, we obtain the gauge fixed action (4.1) in MSFT language:

$$\begin{aligned} S &= -\int d^d\bar{x}\,\mathrm{Tr}\Bigg(\frac{1}{2\alpha'}\int(-d\xi_0)\hat{A}(\bar{x},\xi_0,\xi)\star\hat{\beta}_{\hat{c}_0}(L_0-1)\hat{A}(\bar{x},\xi_0,\xi) \\ &\qquad\qquad + \frac{g}{3}\int d\xi_0^{(3)}d\xi_0^{(2)}d\xi_0^{(1)}\hat{A}\left(\bar{x},\xi_0^{(1)},\xi\right)\star\hat{A}\left(\bar{x},\xi_0^{(2)},\xi\right)\star\hat{A}\left(\bar{x},\xi_0^{(3)},\xi\right)\Bigg) \\ &= -\int d^d\bar{x}\,\mathrm{Tr}\left(\frac{1}{2\alpha'}A(\bar{x},\xi)\star(L_0-1)A(\bar{x},\xi) + \frac{g}{3}A(\bar{x},\xi)\star A(\bar{x},\xi)\star A(\bar{x},\xi)\right)\end{aligned} \tag{4.14}$$

where $\hat{A}\left(\bar{x},\xi_0,\xi\right) = \xi_0\,A(\bar{x},\xi)$ is a Grassmann odd field in the Siegel gauge. The conventional reality condition of the string field in Fock space $\langle V_2|\Psi\rangle = (|\Psi\rangle)^\dagger$ is simply given by the usual reality of the field in Moyal space $A\left(\bar{x},\xi\right)^\dagger = A\left(\bar{x},\xi\right)$.

The equation of motion in MSFT becomes

$$(L_0-1)\,A(\bar{x},\xi) + \alpha' g\,A(\bar{x},\xi)\star A(\bar{x},\xi) = 0, \tag{4.15}$$

which corresponds to the equation of motion in the Siegel gauge $(L_0-1)\Psi + b_0\Psi\star^W\Psi = 0$ in conventional language. The counterpart of the usual classical equation of motion $Q_B\Psi + \Psi\star^W\Psi = 0$ is difficult to express in the cut-off theory as we already commented. Similarly we meet a similar difficulty to express the BRST invariance condition $\tilde{Q}\Psi + b_0c_0\Psi\star^W\Psi = 0$ in the Siegel gauge in MSFT at this stage.

4.2 Computing Feynman graphs including fermionic ghost sector

We have defined the gauge fixed action eq. (4.14) in MSFT language. Based on it, we discuss the Feynman rules in MSFT and show simple examples explicitly. Computations in the matter sector have already been presented in [10].

Vertex. In MSFT the n-string interaction vertex is represented by n-th Moyal $\star$ product and its trace (4.11). In Fourier basis, $e^{i\bar{\xi}\eta}e^{-\bar{\xi}^{gh}\eta^{gh}}$, this amounts to a phase factor to represent the vertex as follows:

$$\begin{aligned}
&\mathrm{Tr}\left((e^{i\bar{\xi}\eta_1}e^{-\bar{\xi}^{gh}\eta_1^{gh}})\star\cdots\star(e^{i\bar{\xi}\eta_n}e^{-\bar{\xi}^{gh}\eta_n^{gh}})\right) = \\
&\quad = \frac{(-)^N(\theta')^{2N}}{(2\pi\theta)^{Nd}}\exp\left(-\frac{1}{2}\sum_{i<j}\bar{\eta}_i\sigma\eta_j - \frac{1}{2}\sum_{i<j}\bar{\eta}_i^{gh}\Sigma\eta_j^{gh}\right)\times \\
&\quad\quad \times(2\pi)^{2Nd}\delta^{2Nd}(\eta_1+\cdots+\eta_n)\,\delta^{4N}(\eta_1^{gh}+\cdots+\eta_n^{gh})\,.
\end{aligned} \tag{4.16}$$

The constant factor comes from $|\det(2\pi\sigma)|^{-d/2}\det\sigma'$ in the definition of the trace.

Propagator. It is convenient to introduce the propagator $\Delta(\eta,\eta',\tau,p)$ in Fourier basis. This was computed in the matter sector in [10]. Here we give the complete form, including the fermionic ghost sector

$$\begin{aligned}
\Delta(\eta,\eta',\tau,p) &:= \int\frac{d^{2Nd}\xi}{(2\pi)^{2Nd}}\,d^{4N}\xi^{gh}\,(e^{-i\bar{\xi}\eta}e^{\bar{\xi}^{gh}\eta^{gh}})\,e^{-\tau L_0(p)}(e^{i\bar{\xi}\eta'}e^{-\bar{\xi}^{gh}\eta'^{gh}}) \\
&= e^{\frac{d-2}{2}\tau\sum_{n>0}\kappa_n+\frac{1}{2}(1+\bar{w}w)l_s^2p^2}\int\frac{d^{2Nd}\xi}{(2\pi)^{2Nd}}\,d^{4N}\xi^{gh}\,e^{-i\bar{\xi}\eta}e^{\bar{\xi}^{gh}\eta^{gh}}\times \\
&\quad\times e^{-\frac{il_s\tau}{2}pw_e\eta_e^x+\frac{\tau}{4}\bar{\eta}'M_0^{-1}\tilde{\kappa}\eta'+\tau\frac{\bar{\partial}}{\partial\eta'}\tilde{\kappa}M_0\frac{\partial}{\partial\eta'}}\,e^{i\bar{\xi}\eta'}\times \\
&\quad\times e^{\frac{\tau}{4}\bar{\eta}'^{gh}\varepsilon M_0^{gh-1}\tilde{\kappa}^{gh}\eta'^{gh}-\tau\frac{\bar{\partial}}{\partial\eta'^{gh}}\varepsilon\tilde{\kappa}^{gh}M_0^{gh}\frac{\partial}{\partial\eta'^{gh}}}\,e^{-\bar{\xi}^{gh}\eta'^{gh}}\,.
\end{aligned} \tag{4.17}$$

Here $L_0(p)$ is given by setting $\beta_0 = l_sp$ in eq. (4.10). Using eqs. (F.5), (F.8), we obtain the propagator in the bc ghost sector

$$\begin{aligned}
\Delta(\eta,\eta',\tau,p) &= g(\tau,p)\,e^{-\bar{\eta}F(\tau)\eta-\bar{\eta}'F(\tau)\eta'+2\bar{\eta}G(\tau)\eta'+(\bar{\eta}+\bar{\eta}')H(\tau,p)}\times \\
&\quad\times e^{\bar{\eta}^{gh}F^{gh}(\tau)\eta^{gh}+\bar{\eta}'^{gh}F^{gh}(\tau)\eta'^{gh}-2\bar{\eta}'^{gh}G^{gh}(\tau)\eta^{gh}}
\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}
g(\tau,p) &= \left(\frac{\theta}{2\pi}\right)^{Nd}\frac{(-1)^N}{\theta'^{2N}}(1+\bar{w}w)^{\frac{d+2}{4}} \\
&\quad\times\left(\prod_{e>0}(1-e^{-2\tau\kappa_e})\prod_{o>0}(1-e^{-2\tau\kappa_o})\right)^{-\frac{d-2}{2}}e^{-\left(\frac{\tau}{2}+\bar{w}\,\frac{\tanh(\frac{\tau\kappa_e}{2})}{\kappa_e}w\right)l_s^2p^2}, \\
F(\tau) &= \frac{1}{4}M_0^{-1}(\tanh(\tau\tilde{\kappa}))^{-1} = \begin{pmatrix}\frac{l_s^2}{2\kappa_e}(\tanh(\tau\kappa_e))^{-1} & 0 \\ 0 & \frac{\theta^2}{8l_s^2}\bar{R}\kappa_o(\tanh(\tau\kappa_o))^{-1}R\end{pmatrix},
\end{aligned}$$

$$
\begin{aligned}
G(\tau) &= \frac{1}{4}M_0^{-1}(\sinh(\tau\tilde{\kappa}))^{-1} = \begin{pmatrix} \frac{l_s^2}{2\kappa_e}(\sinh(\tau\kappa_e))^{-1} & 0 \\ 0 & \frac{\theta^2}{8l_s^2}\bar{R}\kappa_o(\sinh(\tau\kappa_o))^{-1}R \end{pmatrix}, \\
H(\tau,p) &= \frac{\tanh(\tau\kappa_e/2)}{\kappa_e}wl_s^2 p, \\
F^{gh}(\tau) &= \frac{1}{4}\varepsilon M_0^{gh\,-1}(\tanh\tau\tilde{\kappa}^{gh})^{-1} = \varepsilon\begin{pmatrix} -\frac{i}{2}T\kappa_o^{-1}(\tanh\tau\kappa_o)^{-1}\bar{T} & \\ & -\frac{i\theta'^2}{8}\kappa_e(\tanh\tau\kappa_e)^{-1} \end{pmatrix}, \\
G^{gh}(\tau) &= \frac{1}{4}\varepsilon M_0^{gh\,-1}(\sinh\tau\tilde{\kappa}^{gh})^{-1} = \varepsilon\begin{pmatrix} -\frac{i}{2}T\kappa_o^{-1}(\sinh\tau\kappa_o)^{-1}\bar{T} & \\ & -\frac{i\theta'^2}{8}\kappa_e(\sinh\tau\kappa_e)^{-1} \end{pmatrix}.
\end{aligned} \tag{4.19}
$$

The ghost structure of the quadratic term in the exponent is similar to the matter one.

1-loop vacuum amplitude. By taking the trace of the propagator eq. (4.18), we have

$$
\begin{aligned}
\int d^dp\,\mathrm{Tr}\,e^{-\tau(L_0-1)} &= \int d^dp \int d^{2Nd}\eta\, d^{4N}\eta^{gh}\, e^{\tau}\,\Delta(\eta,\eta,p,\tau) \\
&= (2\pi)^{\frac{d}{2}} l_s^{-d}\tau^{-\frac{d}{2}}\, e^{\tau}\prod_{e>0}(1-e^{-\tau\kappa_e})^{-(d-2)}\prod_{o>0}(1-e^{-\tau\kappa_o})^{-(d-2)}.
\end{aligned} \tag{4.20}
$$

This reproduces the expected partition function, with the correct spectrum, including the ghost contribution. If we take the open string limit $\kappa_e = e, \kappa_o = o, N = \infty$ naively in the formula of L_0 (A.47), (2.133), namely at the lagrangian level, we lose the information on odd spectrum. This is one of the indications that the γ term plays a nontrivial role. Of course, we obtain the correct limiting partition function by taking the limit at the last stage of the computation, which is given above.

External state. As external states in Feynman graphs it is enough to consider monoid elements such as

$$
A_{\mathcal{N},M,\lambda,M^{gh},\lambda^{gh}} = \mathcal{N}\, e^{ip\bar{x}} e^{-\bar{\xi}M\xi-\bar{\xi}\lambda} e^{-\bar{\xi}^{gh}M^{gh}\xi^{gh}-\bar{\xi}^{gh}\lambda^{gh}}. \tag{4.21}
$$

In fact, we can compute various perturbative diagrams by preparing a particular class of gaussian external states given by $M = M_0, M^{gh} = \varepsilon M_0^{gh}$, where these matrices were given in eqs.(4.10). If we also take $\bar{\lambda} = (-iw_e p, 0), \lambda^{gh} = 0$, this external field represents perturbative vacuum $\hat{c}_1|p,\Omega\rangle$ with momentum p, which represents the perturbative tachyon with proper normalization

$$
A_p(\xi) = 2^{N(d-2)}(1+\bar{w}w)^{-\frac{d+2}{8}} e^{ip\bar{x}} e^{-\bar{\xi}M_0\xi+ipw_e x_e} e^{-\bar{\xi}^{gh}\varepsilon M_0^{gh}\xi^{gh}}, \qquad \mathrm{Tr}(A_p(\xi)^\dagger \star A_p(\xi)) = 1. \tag{4.22}
$$

We omitted an overall ξ_0 because it drops out in the Siegel gauge action (4.14). Excited states (that correspond to polynomials multiplying this $A_p(\xi)$) can be obtained by differentiating $A_p(\xi)e^{-\bar{\xi}\lambda-\bar{\xi}^{gh}\lambda^{gh}}$ with respect to general λ, λ^{gh} appropriately, and then setting $\lambda = (-iw_e p, 0), \lambda^{gh} = 0$. Therefore an explicit computation of Feynman graphs with general $\lambda, \lambda^{gh}, M, M^{gh}$ has many physical applications.

τ-evolved monoid element. It is convenient to have a τ-evolved formula for a gaussian eq.(4.21) to compute Feynman graphs in ξ-basis [10]. We can derive it explicitly by evaluating:

$$
\begin{aligned}
& e^{-\tau L_0} A_{\mathcal{N},M,\lambda,M^{gh},\lambda^{gh}}(\xi) = \\
& \quad = \int d^{2Nd}\eta\, d^{4N}\eta^{gh}\, e^{-i\bar{\xi}\eta} e^{\bar{\xi}^{gh}\eta^{gh}} \left(\int d^{2Nd}\eta'\, d^{4N}\eta'^{gh} \Delta(\eta,\eta',\tau,p)\, \tilde{A}_{\mathcal{N},M,\lambda,M^{gh},\lambda^{gh}}(\eta') \right), \\
& \tilde{A}_{\mathcal{N},M,\lambda,M^{gh},\lambda^{gh}}(\eta) = \\
& \quad = \int \frac{d^{2Nd}\xi}{(2\pi)^{2Nd}}\, d^{4N}\xi^{gh}\, e^{-i\bar{\xi}\eta} e^{\bar{\xi}^{gh}\eta^{gh}} A_{\mathcal{N},M,\lambda,M^{gh},\lambda^{gh}}(\xi)\,.
\end{aligned} \tag{4.23}
$$

By gaussian integration we have the following formula

$$
e^{-\tau L_0} A_{\mathcal{N},M,\lambda,M^{gh},\lambda^{gh}}(\xi) = \mathcal{N}\mathcal{N}^m(\tau)\mathcal{N}^{gh}(\tau)\, e^{ip\bar{x}} e^{-\bar{\xi}M(\tau)\xi - \bar{\xi}\lambda(\tau)} e^{-\bar{\xi}^{gh}M^{gh}(\tau)\xi^{gh} - \bar{\xi}^{gh}\lambda^{gh}(\tau)} \tag{4.24}
$$

where

$$
M(\tau) = \left[\sinh\tau\tilde{\kappa} + \left(\sinh\tau\tilde{\kappa} + M_0 M^{-1}\cosh\tau\tilde{\kappa}\right)^{-1}\right] (\cosh\tau\tilde{\kappa})^{-1} M_0\,, \tag{4.25}
$$

$$
\lambda(\tau) = \left[\left(\cosh\tau\tilde{\kappa} + M M_0^{-1}\sinh\tau\tilde{\kappa}\right)^{-1}(\lambda + iwp)\right] - iwp\,, \tag{4.26}
$$

$$
\mathcal{N}^m(\tau) = \frac{e^{-\frac{1}{2}l_s^2 p^2\tau}\ \exp\left[\frac{1}{4}\left(\bar{\lambda} + ip\bar{w}\right)(M + \coth\tau\tilde{\kappa}\ M_0)^{-1}(\lambda + iwp)\right]}{\det\left(\frac{1}{2}\left(1 + MM_0^{-1}\right) + \frac{1}{2}\left(1 - MM_0^{-1}\right)e^{-2\tau\tilde{\kappa}}\right)^{d/2}} \tag{4.27}
$$

for the matter sector [10][18] and

$$
M^{gh}(\tau) = \left[\sinh\tau\tilde{\kappa}^{gh} + \left(\sinh\tau\tilde{\kappa}^{gh} + \varepsilon M_0^{gh} M^{gh-1}\cosh\tau\tilde{\kappa}^{gh}\right)^{-1}\right] (\cosh\tau\tilde{\kappa}^{gh})^{-1}\varepsilon M_0^{gh}\,, \tag{4.28}
$$

$$
\lambda^{gh}(\tau) = \left[\cosh\tau\tilde{\kappa}^{gh} - M^{gh}\varepsilon M_0^{gh-1}\sinh\tau\tilde{\kappa}^{gh}\right]^{-1}\lambda^{gh}\,, \tag{4.29}
$$

$$
\begin{aligned}
\mathcal{N}^{gh}(\tau) =& e^{-\frac{1}{4}\bar{\lambda}^{gh}(M^{gh} + \coth(\tau\tilde{\kappa}^{gh})\varepsilon M_0^{gh})^{-1}\lambda^{gh}} \times \\
& \times \left[\det\left(\frac{1}{2}(1 - M^{gh}\varepsilon M_0^{gh-1}) + \frac{1}{2}(1 + M^{gh}\varepsilon M_0^{gh-1})e^{-2\tau\tilde{\kappa}^{gh}}\right)\right]^{\frac{1}{2}}
\end{aligned} \tag{4.30}
$$

for ghost sector. When we consider a class of monoid such that $\bar{\xi}^{gh}M^{gh}\xi^{gh}$ is SU(1,1)-symmetric and twist even[19] then the evolved $M(\tau)$ also has this symmetry. We can see this explicitly by noting that $f(\tilde{\kappa})M_0^{gh}$ is a block diagonal and symmetric matrix (where $f(x)$ is an arbitrary function). In this case with $M^{gh} = \varepsilon M^{gh'}$, where $M^{gh'}$ is a $2N \times 2N$ matrix, the above formula becomes

$$
M^{gh}(\tau) = \varepsilon\left[\sinh\tau\tilde{\kappa}^{gh} + \left(\sinh\tau\tilde{\kappa}^{gh} + M_0^{gh}M^{gh'-1}\cosh\tau\tilde{\kappa}^{gh}\right)^{-1}\right](\cosh\tau\tilde{\kappa}^{gh})^{-1}M_0^{gh}\,, \tag{4.31}
$$

[18] We supposed that M is Lorentz symmetric.

[19] In appendix H, we defined SU(1,1)-symmetric and twist even monoid element which has $\lambda^{gh} = 0$ (H.8). Here we permit $\lambda^{gh} \neq 0$ case.

$$\lambda^{gh}(\tau) = \left[\cosh\tau\tilde{\kappa}^{gh} + M^{gh'} M_0^{gh-1}\sinh\tau\tilde{\kappa}^{gh}\right]^{-1}\lambda^{gh}\,, \tag{4.32}$$

$$\begin{aligned}\mathcal{N}^{gh}(\tau) &= e^{-\frac{1}{4}\bar{\lambda}^{gh}\varepsilon(M^{gh'}+\coth(\tau\tilde{\kappa}^{gh})M_0^{gh})^{-1}\lambda^{gh}}\times \\ &\times\det\left(\frac{1}{2}(1+M^{gh'}M_0^{gh-1})+\frac{1}{2}(1-M^{gh'}M_0^{gh-1})e^{-2\tau\tilde{\kappa}^{gh}}\right)\,.\end{aligned} \tag{4.33}$$

We can use this reduced formula to compute the τ-evolved monoid of n -th product of the perturbative vacuum: $e^{-\tau L_0}\left(A_0(\xi)e^{-\bar{\xi}\lambda_1}\star\cdots\star A_0(\xi)e^{-\bar{\xi}\lambda_n}\right)$ because the coefficient matrix $M_0^{(n)}$ (3.15) in the quadratic term in the exponent is proportional to ε.

With the above preparation, we have all that is needed to compute the ghost contribution in various amplitudes, by following the methods that were developed in the matter sector in [10].

4-tachyon amplitude. The 4 point amplitude for tachyons is computed by putting together several diagrams that are related to each other by permutations of the external legs. For a typical 4-pt diagram ${}^2_1\!>-<^3_4$ the MSFT expression is

$${}_{12}A_{34} = \int d^d\bar{x}\,\mathrm{Tr}\left(e^{-\tau L_0}\left(A_1(\xi)\star A_2(\xi)\right)\star\left(A_3(\xi)\star A_4(\xi)\right)\right) \tag{4.34}$$

where τ is the length of the propagator. When $A_i(\xi)$, $(i=1,\ldots,4)$ are gaussians, we can compute this quantity easily by taking the $\star$ product between the pairs of gaussians (3.4)–(3.9), evolving by τ (4.31), and computing the trace (3.13). At each step we only use the properties of the monoid.

In the case of tachyons, the matter contribution was already computed in [10]. For the ghost contribution, we set the external field to $A_i^{gh}(\xi) = 2^{-2N}(1+\bar{w}w)^{-\frac{1}{4}}\,e^{-\bar{\xi}^{gh}\varepsilon M_0^{gh}\xi^{gh}}$ and obtain

$$\begin{aligned}{}_{12}A_{34}^{gh} &= (\det(2m_0^{gh}))^{-1}(\det(1-(m_0^{gh})^2))^2\times \\ &\times\left[\det\left(4\sinh\tau\tilde{\kappa}^{gh}\left(\left(\cosh\tau\tilde{\kappa}^{gh}+\frac{2}{1+(m_0^{gh})^2}\sinh\tau\tilde{\kappa}^{gh}\right)^2-1\right)^{-1}e^{\tau\tilde{\kappa}^{gh}}\right)\right]^{-1} \\ &= 2^{-8N}(1+\bar{w}w)^{\frac{3}{2}}\,(\det(1+3t\bar{t}))^4\;\times \\ &\times\det\left(1-\left(\frac{t\bar{t}-1}{1+3t\bar{t}}\,e^{-\kappa_e\tau}\right)^2\right)\det\left(1-\left(\frac{\bar{t}t-1}{1+3\bar{t}t}\,e^{-\kappa_o\tau}\right)^2\right)\end{aligned} \tag{4.35}$$

where $m_0^{gh} = M_0^{gh}\sigma'$, $t=\kappa_e^{1/2}T\kappa_o^{-1/2}$. Including the matter sector [10] we obtain

$$\begin{aligned}{}_{12}A_{34} &= 2^{4(d-2)N}(1+\bar{w}w)^{-\frac{d}{4}+\frac{3}{2}}\,(\det(1+3t\bar{t}))^4\,(\det(3+t\bar{t}))^{-2d}\,(2\pi)^d\delta^d(p_1+p_2+p_3+p_4)\times \\ &\times\frac{\det\left(1-\left(\frac{t\bar{t}-1}{1+3t\bar{t}}\,e^{-\kappa_e\tau}\right)^2\right)\det\left(1-\left(\frac{\bar{t}t-1}{1+3\bar{t}t}\,e^{-\kappa_o\tau}\right)^2\right)}{\left[\det\left(1-\left(\frac{t\bar{t}-1}{3+t\bar{t}}\,e^{-\kappa_e\tau}\right)^2\right)\det\left(1-\left(\frac{\bar{t}t-1}{3+\bar{t}t}\,e^{-\kappa_o\tau}\right)^2\right)\right]^{\frac{d}{2}}}\times \\ &\times\exp\left(-\frac{1}{2}l_s^2(p_1+p_2)^2(\tau+\alpha(\tau))+l_s^2(p_1+p_3)^2\beta(\tau)+\frac{1}{2}l_s^2\sum_{i=1}^4 p_i^2\gamma(\tau)\right)\end{aligned} \tag{4.36}$$

where

$$\alpha(\tau) = \bar{v}\kappa_o^{-\frac{1}{2}}\left[\bar{t}\left(1+t\bar{t}+\frac{1}{2}(1+t\bar{t})\coth\frac{\tau\kappa_e}{2}(1+t\bar{t})\right)^{-1}t\right.$$
$$\left.+\left(1+\bar{t}t+\frac{1}{2}(1+\bar{t}t)\coth\frac{\tau\kappa_o}{2}(1+\bar{t}t)\right)^{-1}\right]\kappa_o^{-\frac{1}{2}}v\,, \tag{4.37}$$

$$\beta(\tau) = 2\bar{v}\kappa_o^{-\frac{1}{2}}\Big(4\sinh\tau\kappa_o+(1+\bar{t}t)\,\sinh\tau\kappa_o\,(1+\bar{t}t)$$
$$+2(1+\bar{t}t)\cosh\tau\kappa_o+2\cosh\tau\kappa_o\,(1+\bar{t}t)\Big)^{-1}\kappa_o^{-\frac{1}{2}}v\,, \tag{4.38}$$

$$\gamma(\tau) = -\bar{v}\kappa_o^{-\frac{1}{2}}\coth\frac{\tau\kappa_o}{2}\left(2+(1+\bar{t}t)\coth\frac{\tau\kappa_o}{2}\right)^{-1}\kappa_o^{-\frac{1}{2}}v\,. \tag{4.39}$$

We note that the matrices $t\bar{t}, \bar{t}t$ in the determinant factors in the matter sector come out inverted in the ghost sector. By integrating with the measure $\int_o^\infty d\tau e^\tau$ and adding permutations of the diagram, we should reproduce the Veneziano amplitude[20] when all the tachyons are on-shell, $l_s^2p_i^2 = 2$ in the open string limit $\kappa_e = e, \kappa_o = o, N = \infty$.

Our formula (4.36) has a counterpart in the operator formalism. Although our expressions are simpler it is not easy to compare results analytically because of the different formalisms. We have managed to compare and agree with the determinant factor available in the computations in [29] in the operator formalism, by inserting the MSFT ghost Neumann coefficients given in (3.49) and matter Neumann coefficients taken from [9]

$$\mathcal{M}^{(0)} = \frac{\tilde{m}_0^{*2}-1}{\tilde{m}_0^{*2}+3}\,, \qquad CX^{(0)} = \sqrt{\kappa}\frac{\tilde{m}_0^{*2}-1}{1+3\tilde{m}_0^{*2}}\frac{1}{\sqrt{\kappa}}\,, \qquad \tilde{m}_0^{*2} = \begin{pmatrix} t\bar{t} & 0 \\ 0 & \bar{t}t \end{pmatrix}. \tag{4.40}$$

The difference of normalization compared to [29] comes from that of cubic term of the action (4.13). We note that so far it has not been demonstrated yet that either the operator formalism or the MSFT approach reproduce the Veneziano amplitude analytically, although this is expected to be true.

5. Discussion

In this paper we provided the details of the Moyal star formulation for fermionic ghosts. Following the similar construction in the matter sector, the split string formalism was used as an intermediate step. However, as in the matter sector, the midpoint needed additional considerations to insure that MSFT is in agreement with the operator formulation of string field theory. MSFT then provides an alternative method of computation in string field theory which is in many ways simpler and more efficient.

The regularization of the fermionic ghost sector, which is needed to avoid the associativity anomaly, is made consistently with the matter sector. The correctness of the formulation, including the regularization, was tested by computing the Neumann coefficients by using MSFT methods and comparing them to an independent computation that

[20] The expressions here are slightly further simplified than the ones in [10].

relies on conformal field theory. The MSFT result generalizes the Neumann coefficients by computing them for any set of oscillator frequencies κ_e, κ_o for any finite number of oscillators $2N$. These agree with conformal theory results in the open string limit $\kappa_e = e$, $\kappa_o = o$, $N = \infty$. The agreement was established both analytically as well as numerically.

In numerical study of string field theory one necessarily deals with a finite number of modes. One may debate which version of Neumann coefficients is more consistent for such numerical study: the finite N version of the Neumann coefficients given in MSFT, or the level truncation of the infinite Neumann matrices practiced in previous literature? The numerical analysis that we provided could be helpful in understanding the issue and developing the most appropriate numerical approximation scheme. We hope to address this point, together with numerical studies of certain quantities in the near future.

The regularized MSFT formulation is now complete in the Siegel gauge. It has already been applied to the computation of perturbative Feynman graphs [10] as well as to the analytic study of nonperturbative classical solutions of string theory, including the nonperturbative vacuum of open string theory [12].

An open problem is the construction of a *regularized* version of the BRST operator. The regularization is indispensable to tame the associativity anomaly and to have a well defined theory. Along with the successful regularization in MSFT, the BRST operator is also needed to insure gauge invariance in the general formalism, and to be able to work outside of the Siegel gauge. In particular, the BRST operator can be used to impose the additional gauge invariance conditions in the Siegel gauge on the nonperturbative solutions we have obtained in [12]. Some of the issues surrounding this problem are outlined following eq.(4.8). These remarks apply not only to MSFT, but also to any version of string theory that uses a cutoff of the string modes (including level truncation), since the Virasoro algebra does not close with a finite number of modes. A substitute for the Virasoro algebra at finite N, which tends to the Virasoro algebra at infinite N, is the key to solving this problem.

Acknowledgments

I.K. would like to thank H. Hata, T. Kawano and K. Ohmori for valuable discussions and comments. I.B. is supported in part by a DOE grant DE-FG03-84ER40168. I.K. is supported in part by JSPS Research Fellowships for Young Scientists. Y.M. is supported in part by Grant-in-Aid (# 13640267) from the Ministry of Education, Science, Sports and Culture of Japan.

A. Brief review of MSFT in matter sector

A.1 Half-string for cosine modes

Here we review the split string formulation and its regularization which was developed in [8] and fix notation in this paper. Although it was constructed to formulate the matter and the bosonized ghost sector, we can apply the same formalism to fermionic functions which have a Fourier expansion in terms of cosine mode in the full string basis.

JHEP07(2003)027

A full string function $\phi(\sigma)$ which satisfies Neumann boundary conditions at the end points

$$\frac{d}{d\sigma}\phi(\sigma)\Big|_{\sigma=0} = \frac{d}{d\sigma}\phi(\sigma)\Big|_{\sigma=\pi} = 0\,, \tag{A.1}$$

has a Fourier expansion in terms of cosine modes

$$\phi(\sigma) = \phi_0 + \sqrt{2}\sum_{n=1}^{\infty}\phi_n \cos n\sigma\,, \qquad \phi_0 = \frac{1}{\pi}\int_0^{\pi} d\sigma\phi(\sigma)\,, \quad \phi_n = \frac{\sqrt{2}}{\pi}\int_0^{\pi} d\sigma\phi(\sigma)\cos n\sigma\,. \tag{A.2}$$

Then split string functions $l(\sigma), r(\sigma)$ for $\phi(\sigma)$ are defined as

$$\phi(\sigma) = \begin{cases} l(\sigma) & (0 \le \sigma \le \frac{\pi}{2}) \\ r(\pi-\sigma) & (\frac{\pi}{2} \le \sigma \le \pi) \end{cases}\,, \tag{A.3}$$

$$\phi_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\sigma(l(\sigma)+r(\sigma))\,, \qquad \phi_n = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\sigma(l(\sigma)+(-1)^n r(\sigma))\cos n\sigma\,. \tag{A.4}$$

A.1.1 Dirichlet at midpoint

Split string functions with Neumann boundary conditions at the end points and Dirichlet boundary conditions at the midpoint

$$l'(0) = r'(0) = 0\,, \qquad l\left(\frac{\pi}{2}\right) = r\left(\frac{\pi}{2}\right) = \bar{\phi}\left(:= \phi\left(\frac{\pi}{2}\right)\right)\,, \tag{A.5}$$

have a Fourier expansion in terms of odd cosine modes $o = 1, 3, 5, \ldots$

$$l(\sigma) = \bar{\phi} + \sqrt{2}\sum_{o=1}^{\infty} l_o \cos o\sigma\,, \qquad r(\sigma) = \bar{\phi} + \sqrt{2}\sum_{o=1}^{\infty} r_o \cos o\sigma\,. \tag{A.6}$$

The correspondence between $\{\bar{\phi}, l_o, r_o\}$ and $\{\phi_e, \phi_o\}$ is

$$\bar{\phi} = \phi_0 - \bar{w}\phi_e\,, \qquad l_o = \phi_o + R\phi_e\,, \qquad r_o = -\phi_o + R\phi_e\,, \tag{A.7}$$

$$\phi_0 = \bar{\phi} + \frac{1}{2}\bar{v}(l_o+r_o)\,, \qquad \phi_e = \frac{1}{2}T(l_o+r_o)\,, \qquad \phi_o = \frac{1}{2}(l_o-r_o)\,, \tag{A.8}$$

where we used matrix notation for simplicity, and denoted

$$\begin{aligned} R_{oe} &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} d\sigma \cos o\sigma\left(\cos e\sigma - \cos\frac{e\pi}{2}\right) = \frac{4e^2\, i^{o-e+1}}{\pi o(e^2-o^2)}\,, \\ T_{eo} &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} d\sigma \cos e\sigma \cos o\sigma = \frac{4o\, i^{o-e+1}}{\pi(e^2-o^2)}\,, \\ v_o &= \frac{2\sqrt{2}\, i^{o-1}}{\pi o} = \frac{1}{\sqrt{2}}T_{0o}\,, \qquad w_e = \sqrt{2}\, i^{-e+2}\,. \end{aligned} \tag{A.9}$$

A.1.2 Neumann at midpoint

Split string functions with Neumann boundary conditions at the end points and Neumann boundary conditions at the midpoint

$$l'(0) = r'(0) = 0\,, \qquad l'\left(\frac{\pi}{2}\right) = r'\left(\frac{\pi}{2}\right) = 0\,, \tag{A.10}$$

JHEP07(2003)027

are expanded in terms of even cosine modes $e = 2, 4, 6, \ldots$

$$l(\sigma) = \bar{\phi} + \sqrt{2}\sum_{e=2}^{\infty} l_e \left(\cos e\sigma - i^e\right) , \quad r(\sigma) = \bar{\phi} + \sqrt{2}\sum_{e=2}^{\infty} r_e \left(\cos e\sigma - i^e\right) . \tag{A.11}$$

The correspondence between split string and full string modes is

$$\bar{\phi} = \phi_0 - \bar{w}\phi_e , \qquad l_e = \phi_e + T\phi_o , \qquad r_e = \phi_e - T\phi_o , \tag{A.12}$$

$$\phi_0 = \bar{\phi} + \frac{1}{2}\bar{w}(l_e + r_e) , \qquad \phi_e = \frac{1}{2}(l_e + r_e) , \qquad \phi_o = \frac{1}{2}R(l_e - r_e) . \tag{A.13}$$

A.1.3 Regularization

The infinite matrices T, R and vectors v, w defined in eq.(A.10) satisfy the following relations

$$R_{oe} = o^{-2} T_{eo} e^2 , \qquad R_{oe} = T_{eo} + v_o w_e , \qquad v_o = \sum_{e>0} T_{eo} w_e , \qquad w_e = \sum_{o>0} R_{oe} v_e . \tag{A.14}$$

As noted in [8], there is an ambiguity in naive computation using these matrices and vectors. For example, T has an inverse matrix given by R, and yet it has a zero eigenvalue $Tv = 0$. This is possible only because they are infinite dimensional matrices. It causes associativity anomalies. To avoid the ambiguous results that come from the associativity anomaly, a finite matrix regularization is proposed in [8]. We define $N \times N$ matrices T, R and N-vectors v, w by

$$R_{oe} = (\kappa_o)^{-2}\, \bar{T}_{oe}\, (\kappa_e)^2 , \qquad R_{oe} = \bar{T}_{oe} + v_o \bar{w}_e , \qquad v_o = \bar{T}_{oe} w_e , \qquad w_e = \bar{R}_{eo} v_o , \tag{A.15}$$

where a bar means transpose, and we introduced a set of $2N$ frequencies κ_e, κ_o. These relations are identical to the ones satisfied by the infinite matrices in eq. (A.14), but we now use them as defining relations for finite dimensional matrices and arbitrary frequencies. We can solve the equations in eq. (A.15) explicitly in term of the frequencies

$$T_{eo} = \frac{w_e v_o \kappa_o^2}{\kappa_e^2 - \kappa_o^2} , \qquad R_{oe} = \frac{w_e v_o \kappa_e^2}{\kappa_e^2 - \kappa_o^2} , \tag{A.16}$$

$$w_e = i^{2-e} \frac{\prod_{o'} \left|\kappa_e^2/\kappa_{o'}^2 - 1\right|^{\frac{1}{2}}}{\prod_{e'\neq e} \left|\kappa_e^2/\kappa_{e'}^2 - 1\right|^{\frac{1}{2}}} , \qquad v_o = i^{o-1} \frac{\prod_{e'} \left|1 - \kappa_o^2/\kappa_{e'}^2\right|^{\frac{1}{2}}}{\prod_{o'\neq o} \left|1 - \kappa_o^2/\kappa_{o'}^2\right|^{\frac{1}{2}}} . \tag{A.17}$$

By using only the defining relations we can show the following further relations for the regularized version of T, R, v, w.

$$\begin{aligned} TR &= 1_e , \qquad RT = 1_o , \qquad \bar{R}R = 1 + w\bar{w} , \qquad \bar{T}T = 1 - v\bar{v} , \\ T\bar{T} &= 1 - \frac{w\bar{w}}{1 + \bar{w}w} , \qquad Tv = \frac{w}{1 + \bar{w}w} , \qquad \bar{v}v = \frac{\bar{w}w}{1 + \bar{w}w} , \\ Rw &= v(1 + \bar{w}w) , \qquad R\bar{R} = 1 + v\bar{v}\,(1 + \bar{w}w) . \end{aligned} \tag{A.18}$$

The original T, R, v, w in eq. (A.10) are reproduced by setting the open string limit:

$$\kappa_e = e , \qquad \kappa_o = o , \qquad N \to \infty . \tag{A.19}$$

We note that at this limit $\bar{w}w$ diverges

$$1+\bar{w}w=\left(\prod_{n=1}^{N}\frac{\kappa_{2n}}{\kappa_{2n-1}}\right)^{2} \quad\rightarrow\quad \left(\frac{\sqrt{\pi}\,\Gamma\left(N+1\right)}{\Gamma\left(N+\frac{1}{2}\right)}\right)^{2}\rightarrow\ \infty\,. \tag{A.20}$$

A.2 Some results in matter sector

Here we summarize notation and conventions in the matter sector in MSFT.[21]

A.2.1 Oscillators in MSFT

- Mode expansion in matter sector ($\mu=0,1,\ldots,d-1$):

$$X^{\mu}(\sigma)\,=\,\hat{x}_0^{\mu}+\sqrt{2}\sum_{n=1}^{\infty}\hat{x}_n^{\mu}\cos n\sigma=\hat{x}_0^{\mu}+i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_n^{\mu}-\alpha_{-n}^{\mu})\cos n\sigma\,, \tag{A.21}$$

$$P_{\mu}(\sigma)\,=\,\frac{1}{\pi}\left(\hat{p}_{0\mu}+\sqrt{2}\sum_{n=1}^{\infty}\hat{p}_{n\mu}\cos n\sigma\right)=\frac{1}{\pi}\left(\hat{p}_{0\mu}+\frac{1}{\sqrt{2\alpha'}}\sum_{n=1}^{\infty}\eta_{\mu\nu}(\alpha_n^{\nu}+\alpha_{-n}^{\nu})\cos n\sigma\right).$$

Nonzero modes in matter sector

$$\begin{aligned}
&\alpha_n^{\mu}=\sqrt{\kappa_n}\hat{a}_n^{\mu}\,, \qquad \alpha_{-n}^{\mu}=\sqrt{\kappa_n}\hat{a}_n^{\dagger\mu}\,,\\
&[\alpha_n^{\mu},\alpha_m^{\nu}]=\epsilon(n)\kappa_n\delta_{n+m,0}\eta^{\mu\nu}\,, \qquad [\hat{a}_n^{\mu},\hat{a}_m^{\dagger\nu}]=\delta_{n,m}\eta^{\mu\nu}\,,\\
&x_n^{\mu}=\frac{i}{\sqrt{2\kappa_n}}l_s(\hat{a}_n^{\mu}-\hat{a}_n^{\dagger\mu})\,, \qquad p_{n\mu}=\sqrt{\frac{\kappa_n}{2}}\frac{\eta_{\mu\nu}}{l_s}(\hat{a}_n^{\nu}+\hat{a}_n^{\dagger\nu})\,,\\
&[\hat{x}_n^{\mu},\hat{p}_{m\nu}]=i\delta_{n,m}\delta_{\nu}^{\mu}\,, \qquad \alpha_n=\frac{1}{\sqrt{2}}\left(l_s\hat{p}_{|n|}-i\epsilon(n)\frac{\kappa_{|n|}}{l_s}\hat{x}_{|n|}\right)\,,
\end{aligned} \tag{A.22}$$

where we define the symbols $l_s^2=2\alpha',\kappa_n=n$.

Zero mode in matter sector, with $a_0^{\mu}:=l_s\hat{p}_0^{\mu}$

$$\hat{x}_0^{\mu}=\frac{i}{2}l_s\sqrt{b}(\hat{a}_0^{\mu}-\hat{a}_0^{\dagger\mu})\,,\quad \hat{p}_{0\mu}=\frac{1}{l_s\sqrt{b}}\eta_{\mu\nu}(\hat{a}_0^{\nu}+\hat{a}_0^{\dagger\nu})\,,\quad [\hat{x}_0^{\mu},\hat{p}_{0\nu}]=i\delta_{\nu}^{\mu}\,,\quad [\hat{a}_0^{\mu},\hat{a}_0^{\dagger\nu}]=\eta^{\mu\nu} \tag{A.23}$$

where b is some positive constant.

- Position eigenstates

$$\begin{aligned}
\langle x_0,x_n|\hat{x}_n\,&=\,\langle x_0,x_n|x_n\,, \qquad \langle x_0,x_n|\hat{x}_0=\langle x_0,x_n|x_0\,,\\
\hat{x}_n|x_0,x_n\rangle\,&=\,x_n|x_0,x_n\rangle\,, \qquad \hat{x}_0|x_0,x_n\rangle=x_0|x_0,x_n\rangle,
\end{aligned} \tag{A.24}$$

are given as squeezed states in Fock space

$$\langle x_0,x_n|\,=\,\langle x_0|e^{\sum_{n>0}\left(\frac{1}{2\kappa_n}\alpha_n^2+\frac{i\sqrt{2}}{l_s}x_n\alpha_n-\frac{\kappa_n}{2l_s^2}x_n^2\right)}\prod_{n>0}\left(\frac{\kappa_n}{\pi l_s^2}\right)^{\frac{d}{4}}$$

[21] In this subsection, we use the same symbols for matter as we did for the ghosts in the main text for some quantities, such as positions and momenta. We can avoid confusion from the context. We omit some definitions and details because we can refer to ref. [9] for them.

JHEP07(2003)027

$$
\begin{aligned}
&= \langle x_0| e^{\sum_{n>0}\left(\frac{1}{2}\hat{a}_n^2+\frac{i\sqrt{2\kappa_n}}{l_s}x_n\hat{a}_n-\frac{\kappa_n}{2l_s^2}x_n^2\right)}\prod_{n>0}\left(\frac{\kappa_n}{\pi l_s^2}\right)^{\frac{d}{4}},\\
\langle x_0| &= \langle 0| e^{\frac{1}{2}\hat{a}_0^2+i\frac{2}{l_s\sqrt{b}}x_0\hat{a}_0-\frac{1}{l_s^2 b}x_0^2}\left(\frac{2}{\pi l_s^2 b}\right)^{\frac{d}{4}},\\
|x_0,x_n\rangle &= \prod_{n>0}\left(\frac{\kappa_n}{\pi l_s^2}\right)^{\frac{d}{4}} e^{\sum_{n>0}\left(\frac{1}{2\kappa_n}\alpha_{-n}^2-\frac{i\sqrt{2}}{l_s}x_n\alpha_{-n}-\frac{\kappa_n}{2l_s^2}x_n^2\right)}|x_0\rangle\\
&= \prod_{n>0}\left(\frac{\kappa_n}{\pi l_s^2}\right)^{\frac{d}{4}} e^{\sum_{n>0}\left(\frac{1}{2}\hat{a}_n^{\dagger 2}-\frac{i\sqrt{2\kappa_n}}{l_s}x_n\hat{a}_n^\dagger-\frac{\kappa_n}{2l_s^2}x_n^2\right)}|x_0\rangle\,,\\
|x_0\rangle &= \left(\frac{2}{\pi l_s^2 b}\right)^{\frac{d}{4}} e^{\frac{1}{2}\hat{a}_0^{\dagger 2}-i\frac{2}{l_s\sqrt{b}}x_0\hat{a}_0^\dagger-\frac{1}{l_s^2 b}x_0^2}|0\rangle\,.
\end{aligned}
\tag{A.25}
$$

They satisfy normalization and completeness conditions

$$
\begin{gathered}
\langle x_0,x_n|x_0',x_n'\rangle = \delta^d(x_0-x_0')\prod_{n>0}\delta^d(x_n-x_n')\,,\\
\int d^dx_0\prod_{n>0}d^dx_n|x_0,x_n\rangle\langle x_0,x_n| = 1\,.
\end{gathered}
\tag{A.26}
$$

- Oscillators as differential operators in position space

$$
\begin{aligned}
\langle x_0,x_n|\alpha_m|\Psi\rangle &= -\frac{i}{\sqrt{2}}\left(\epsilon(m)\frac{\kappa_{|m|}}{l_s}x_{|m|}+l_s\frac{\partial}{\partial x_{|m|}}\right)\langle x_0,x_n|\Psi\rangle\,,\\
\langle x_0,x_n|\alpha_0|\Psi\rangle &= -il_s\frac{\partial}{\partial x_0}\langle x_0,x_n|\Psi\rangle\,.
\end{aligned}
\tag{A.27}
$$

- Transformation from position space to Moyal space

$$
\begin{aligned}
A(\bar{x},x_e,p_e) &= (\det(2T))^{d/2}\int dx_o e^{-\frac{2i}{\theta}p_eTx_o}\Psi(x_0,x_n)\\
&= 2^{\frac{Nd}{2}}(1+\bar{w}w)^{-\frac{d}{4}}\int dx_o e^{-\frac{2i}{\theta}p_eTx_o}\langle x_0,x_n|\Psi\rangle =: \langle\bar{x},x_e,p_e|\Psi\rangle\,,
\end{aligned}
\tag{A.28}
$$

where the midpoint $\bar{x}^\mu := X^\mu(\pi/2)$ is related to the center of mass x_0 through eq. (A.21)

$$
\bar{x}^\mu := X^\mu\left(\frac{\pi}{2}\right) = x_0^\mu-\sum_e x_e^\mu w_e\,,\qquad \langle\bar{x}| = \langle x_0|\exp\left(i\hat{p}\cdot\sum_e x_e w_e\right).
\tag{A.29}
$$

The Moyal space $\langle\bar{x},x_e,p_e|$ is given by a squeezed state

$$
\begin{aligned}
\langle\bar{x},x_e,p_e| &= \langle\bar{x}|e^{\frac{\alpha_e^2}{2\kappa_e}-\frac{\alpha_o^2}{2\kappa_o}-\bar{\xi}M_0\xi-\bar{\xi}\lambda}\det(4{\kappa_e}^{1/2}T\kappa_o^{-1/2})^{\frac{d}{2}}\\
&= \langle\bar{x}|e^{\frac{\alpha_e^2}{2\kappa_e}-\frac{\alpha_o^2}{2\kappa_o}-\bar{\xi}M_0\xi-\bar{\xi}\lambda}\,2^{Nd}(1+\bar{w}w)^{-\frac{d}{8}}\,,\\
M_0 &= \begin{pmatrix}\frac{\kappa_e}{2l_s^2} & 0\\ 0 & \frac{2l_s^2}{\theta^2}T\kappa_o^{-1}\bar{T}\end{pmatrix},\qquad \lambda = \begin{pmatrix}-i\frac{\sqrt{2}}{l_s}\alpha_e-i\hat{p}w_e\\ -\frac{2\sqrt{2}l_s}{\theta}T\kappa_o^{-1}\alpha_o\end{pmatrix}.
\end{aligned}
\tag{A.30}
$$

- Moyal $\star$ product and trace:

$$\star = \exp\left(\frac{1}{2}\overleftarrow{\partial_\xi}\sigma\overrightarrow{\partial_\xi}\right) = \exp\left(\frac{i\theta}{2}\left(\overleftarrow{\partial_{x_e}}\overrightarrow{\partial_{p_e}} - \overleftarrow{\partial_{p_e}}\overrightarrow{\partial_{x_e}}\right)\right), \quad \sigma = i\theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.31}$$

$$\mathrm{Tr}\, A(\bar{x},\xi) = |\det(2\pi\sigma)|^{-\frac{d}{2}}\int dx_e dp_e A(\bar{x},\xi) = (2\pi\theta)^{-Nd}\int dx_e dp_e A(\bar{x},\xi)\,. \tag{A.32}$$

The normalization of a field A in Moyal space coincides with the normalization of its image in Fock space (A.28)

$$\langle\Psi|\Psi\rangle = \int d^d\bar{x}\,\mathrm{Tr}\left(A^\dagger(\bar{x},\xi)\star A(\bar{x},\xi)\right)\,. \tag{A.33}$$

- Oscillators as differential operators in Moyal space

$$\begin{aligned}
\langle\bar{x},x_e,p_e|\alpha_0|\Psi\rangle &= -il_s\frac{\partial}{\partial\bar{x}}\langle\bar{x},x_e,p_e|\Psi\rangle =: \beta_0\langle\bar{x},x_e,p_e|\Psi\rangle\,,\\
\langle\bar{x},x_e,p_e|\alpha_e|\Psi\rangle &= \left(\bar{\beta}^x_e - \frac{w_{|e|}}{\sqrt{2}}\beta_0\right)\langle\bar{x},x_e,p_e|\Psi\rangle = \left(\bar{\beta}^x_e - w'_e\beta_0\right)\langle\bar{x},x_e,p_e|\Psi\rangle\\
&= \beta^x_e\langle\bar{x},x_e,p_e|\Psi\rangle\,,\\
\langle\bar{x},x_e,p_e|\alpha_o|\Psi\rangle &= :\beta^p_o\langle\bar{x},x_e,p_e|\Psi\rangle = \sum_{e\neq 0}\bar{\beta}^p_e\langle\bar{x},x_e,p_e|\Psi\rangle U_{-e,o}\,,\\
\beta^p_o &= \sum_{e>0}\frac{1}{\sqrt{2}}\left(\epsilon(o)\frac{\theta\kappa_{|o|}}{2l_s}R_{|o|e}\frac{\partial}{\partial p_e} + \frac{2l_s}{\theta}p_e T_{e|o|}\right) = \sum_{e\neq 0}\bar{\beta}^p_e U_{-e,o}\,,\\
\bar{\beta}^x_e &= -\frac{i}{\sqrt{2}}\left(\epsilon(e)\frac{\kappa_{|e|}}{l_s}x_{|e|} + l_s\frac{\partial}{\partial x_{|e|}}\right),\\
\bar{\beta}^p_e &= \frac{1}{\sqrt{2}}\left(\frac{\theta\kappa_{|e|}}{2l_s}\epsilon(e)\frac{\partial}{\partial p_{|e|}} + \frac{2l_s}{\theta}p_{|e|}\right).
\end{aligned} \tag{A.34}$$

These satisfy ordinary commutation relation:

$$\begin{aligned}
&[\bar{\beta}^x_e,\bar{\beta}^x_{e'}] = \epsilon(e)\kappa_{|e|}\delta_{e+e'}\,, \qquad [\bar{\beta}^p_e,\bar{\beta}^p_{e'}] = \epsilon(e)\kappa_{|e|}\delta_{e+e'}\,, \qquad [\bar{\beta}^x_e,\bar{\beta}^p_{e'}] = 0\,,\\
&[\beta^x_e,\beta^x_{e'}] = \epsilon(e)\kappa_{|e|}\delta_{e+e'}\,, \qquad [\beta^p_o,\beta^p_{o'}] = \epsilon(o)\kappa_{|o|}\delta_{o+o'}\,, \qquad [\beta^x_e,\beta^p_{o'}] = 0\,.
\end{aligned} \tag{A.35}$$

- Oscillators as fields in Moyal space[22]

$$\begin{aligned}
\bar{\beta}^x_e A &= \sqrt{\frac{\kappa_{|e|}}{2}}\left(\beta_e\star A - A\star\beta_{-e}\right), \qquad \bar{\beta}^p_e A = \sqrt{\frac{\kappa_{|e|}}{2}}\left(\beta_e\star A + A\star\beta_{-e}\right),\\
\beta_e :&= \frac{1}{\sqrt{\kappa_{|e|}}}\left(-\frac{i}{2l_s}\epsilon(e)\kappa_{|e|}x_{|e|} + \frac{l_s}{\theta}p_{|e|}\right),\\
[\beta_e,\beta_{e'}]_\star &= \epsilon(e)\delta_{e+e'}\,.
\end{aligned} \tag{A.36}$$

We can also define odd mode fields through a Bogoliubov transformation

$$\sqrt{\kappa_{|o|}}\beta_o := \sum_{e\neq 0}\sqrt{\kappa_{|e|}}\beta_e U_{-e,o}\,. \tag{A.37}$$

[22]The convention for β_n here is the same as [12] $\beta^{\mathrm{BKM}}_n = \beta_n$, but differs by a factor from the convention in [9] $\beta^{\mathrm{BM}}_n = \sqrt{\frac{\kappa_{|n|}}{2}}\beta_n$.

The following relations hold

$$\beta^p_o A = \sqrt{\frac{\kappa_{|o|}}{2}}\,(\beta_o \star A + A \star \beta_{-o}) \ ,$$
$$[\beta_o, \beta_{o'}]_\star = \epsilon(o)\delta_{o+o'} \ , \qquad [\beta_{-e}, \beta_o]_\star = -\epsilon(e)\kappa^{\frac{1}{2}}_{|e|} U_{-e,o} \kappa^{\frac{1}{2}}_{|o|} \, . \tag{A.38}$$

A.2.2 Butterfly projector

The momentum independent butterfly state $A_B(\xi)$ satisfies

$$\beta_e \star A_B = A_B \star \beta_{-e} = 0 \, , \qquad \forall e > 0 \, . \tag{A.39}$$

There is a unique solution in monoid

$$A_B(\xi) = 2^{dN} \exp\left(- \sum_{e>0} \left(\frac{1}{2l_s^2} \bar{x}_e \kappa_e x_e + \frac{2l_s^2}{\theta^2} \bar{p}_e \frac{1}{\kappa_e} p_e \right) \right) . \tag{A.40}$$

It satisfies

$$A_B \star A_B = A_B \, , \qquad Tr\,(A_B) = 1 \, . \tag{A.41}$$

In ordinary oscillator language, eq. (A.39) means

$$\alpha_e |\Psi_B\rangle = 0 \, , \quad \sum_{o>0} \left(\alpha_o U^{-1}_{-o,e} + \alpha_{-o} U^{-1}_{o,e} \right) |\Psi_B\rangle = 0 \, , \qquad \forall e > 0 \tag{A.42}$$

for zero momentum state. Now we take the ansatz

$$|\Psi_B\rangle = \mathcal{N} \exp\left(-\frac{1}{2} \sum_{m,n \geq 1} a^\dagger_m V^B_{mn} a^\dagger_n \right) |\Omega\rangle \tag{A.43}$$

which corresponds to a monoid element in MSFT. Then we have constraints for V^B_{mn} :

$$\sum_{o'>0} V^B_{oo'} \sqrt{\kappa_{o'}} U^{-1}_{-o',e} = \sqrt{\kappa_o} U^{-1}_{o,e} \, , \qquad o, e > 0 \, , \qquad V^B_{eo} = V^B_{oe} = V^B_{ee'} = 0 \, . \tag{A.44}$$

At the open string limit $\kappa_e = e, \kappa_o = o, N = \infty$, we can show that the matrix V^B_{mn} which was obtained in [27]:

$$V^B_{mn} = \begin{cases} -(-1)^{\frac{m+n}{2}} \frac{\sqrt{mn}}{m+n} \frac{\Gamma[\frac{m}{2}]\Gamma[\frac{n}{2}]}{\pi \Gamma[\frac{m+1}{2}]\Gamma[\frac{n+1}{2}]} & \text{for } m \text{ and } n \text{ odd} \\ 0 & \text{for } m \text{ or } n \text{ even} \end{cases} \tag{A.45}$$

satisfies eq. (A.44). Namely, we have obtained the correspondence:

$$A_B \quad \leftrightarrow \quad |\Psi_B\rangle = \exp\left(-\frac{1}{2} L_{-2} \right) |\Omega\rangle \, , \qquad \text{for} \quad \kappa_e = e, \kappa_o = o, N = \infty \, . \tag{A.46}$$

A.2.3 L_0 and $\mathcal{L}_0$

In MSFT L_0 in matter sector is defined as

$$\begin{aligned}L_0 &= \frac{1}{2}\beta_0^2 + \sum_{e>0}\beta^x_{-e}\beta^x_e + \sum_{o>0}\beta^p_{-o}\beta^p_o \\ &= \frac{1}{2}(1+\bar{w}w)\beta_0^2 + \beta_0\sum_{e>0} il_s w_e \frac{\partial}{\partial x_e} - \frac{d}{2}\sum_{n>0}\kappa_n \qquad \text{(A.47)} \\ &\quad + \sum_{e>0}\left(-\frac{l_s^2}{2}\frac{\partial^2}{\partial x_e^2} - \frac{\theta^2}{8l_s^2}\kappa_e^2\frac{\partial^2}{\partial p_e^2} + \frac{1}{2l_s^2}\kappa_e^2 x_e^2 + \frac{2l_s^2}{\theta^2}p_e^2\right) - \frac{1}{1+\bar{w}w}\frac{2l_s^2}{\theta^2}\left(\sum_{e>0} w_e p_e\right)^2 .\end{aligned}$$

The operator L_0 can be rewritten in terms of a field $\mathcal{L}_0$ using Moyal star product, plus a remnant γ called the "midpoint correction term" [12], which is multiplied with an ordinary product

$$L_0 A_{\beta_0} = \mathcal{L}_0(\beta_0)\star A_{\beta_0} + A_{\beta_0}\star\mathcal{L}_0(-\beta_0) + \gamma A_{\beta_0}\,, \qquad \text{(A.48)}$$

$$\mathcal{L}_0(\beta_0) := \sum_{e>0}\left(\frac{l_s^2}{\theta^2}p_e^2 + \frac{\kappa_e^2}{4l_s^2}x_e^2 - \frac{l_s}{\theta}w_e p_e\beta_0\right) + \frac{1}{4}(1+\bar{w}w)\beta_0^2 - \frac{d}{4}\sum_{n>0}\kappa_n\,, \qquad \text{(A.49)}$$

$$\gamma = -\frac{1}{1+\bar{w}w}\frac{2l_s^2}{\theta^2}\left(\sum_{e>0} w_e p_e\right)^2 . \qquad \text{(A.50)}$$

The γ term *formally* goes to zero as $\kappa_n = n, N\to\infty$, since $\bar{w}w\to\infty$. However, this is not true in computations due to contributions of the form ∞/∞ that are related to the associativity anomaly. In fact, γ is indispensable to reproduce the correct spectrum of L_0 [9][10]. The γ term depends only on one special momentum mode $\hat{p} = (1+\bar{w}w)^{-1/2}\sum_{e>0} w_e p_e$ which we call the anomalous midpoint momentum mode [9, 11]. We can rewrite $\mathcal{L}_0$ in terms of oscillators

$$\mathcal{L}_0(\beta_0) = \sum_{e>0}\kappa_e\beta_{-e}\star\beta_e + \frac{d}{4}\left(\sum_{e>0}\kappa_e - \sum_{o>0}\kappa_o\right) + \frac{1}{4}(1+\bar{w}w)\beta_0^2 - \frac{l_s}{\theta}\left(\bar{w}_e p_e\right)\beta_0\,, \qquad \text{(A.51)}$$

and then, acting on the butterfly projector (A.39), we have

$$L_0 A_B = \gamma A_B\,. \qquad \text{(A.52)}$$

A.2.4 n-string vertex and Neumann coefficients

Here we give brief review of the correspondence of n-string vertex and Neumann coefficients in MSFT. We note the properties of the momentum state and coherent states in Fock space, together with the corresponding Moyal images.

- Momentum eigenstate (zero mode part):

$$\begin{aligned}|p_0\rangle &= \left(2\pi b l_s^2\right)^{\frac{d}{4}} e^{-\frac{1}{2}\hat{a}_0^{\dagger 2} + \sqrt{b}l_s\hat{a}_0^\dagger p_0 - \frac{bl_s^2}{4}p_0^2}|0\rangle\,, \\ \langle p_0| &= \langle 0|e^{-\frac{1}{2}\hat{a}_0^2 + \sqrt{b}l_s\hat{a}_0 p_0 - \frac{bl_s^2}{4}p_0^2}\left(2\pi b l_s^2\right)^{\frac{d}{4}}\,, \\ \langle p_0|p_0'\rangle &= (2\pi)^d\delta^d(p_0 - p_0')\,, \\ \langle p_0|x_0\rangle &= e^{-ip_0x_0}\,, \qquad \langle x_0|p_0\rangle = e^{ip_0x_0}\,. \qquad \text{(A.53)}\end{aligned}$$

- Coherent state:[23]

$$\begin{aligned}
\langle\Psi|\hat{a}^\dagger &= \langle\Psi|\mu^*\,, \qquad \langle\Psi| = \langle p|e^{\mu^*\hat{a}}\,,\\
{}_2\langle\Psi|V_2\rangle_{12} &= e^{-\mu^* C\hat{a}^{\dagger(1)}}|-p\rangle_1 =: |\tilde{\Psi}\rangle_1\,,\\
\tilde{A} :&= \langle\bar{x},x_e,p_e|\tilde{\Psi}\rangle = 2^{Nd}(1+\bar{w}w)^{-\frac{d}{8}}e^{-i\bar{x}p}e^{\frac{1}{2}\mu_e^{*2}-\frac{1}{2}\mu_o^{*2}-\bar{\xi}M_0\xi-\bar{\xi}\lambda}\,,\\
\lambda &= \begin{pmatrix}\frac{i\sqrt{2}}{l_s}\kappa_e^{\frac{1}{2}}\mu_e^* + ipw_e\\ -\frac{2\sqrt{2}l_s}{\theta}T\kappa_o^{-\frac{1}{2}}\mu_o^*\end{pmatrix} = 2K^*(\mu^*+Wp)\,,\\
K^* &= \begin{pmatrix}\frac{i}{l_s}\sqrt{\frac{\kappa_e}{2}} & 0\\ 0 & -\frac{l_s}{\theta}T\sqrt{\frac{2}{\kappa_o}}\end{pmatrix}\,, \qquad W = \begin{pmatrix}\frac{l_s}{\sqrt{2\kappa_e}}w\\ 0\end{pmatrix}\,,
\end{aligned} \tag{A.54}$$

where we used the reflector

$$\begin{aligned}
\langle V_2| &= \int\frac{d^dp^{(1)}}{(2\pi)^d}\frac{d^dp^{(2)}}{(2\pi)^d}\langle 0,p^{(1)}|\langle 0,p^{(2)}|e^{-\sum_{n\geq1}(-1)^n\hat{a}_n^{(1)}\hat{a}_n^{(2)}}(2\pi)^d\delta^d(p^{(1)}+p^{(2)})\,,\\
|V_2\rangle &= \int\frac{d^dp^{(1)}}{(2\pi)^d}\frac{d^dp^{(2)}}{(2\pi)^d}(2\pi)^d\delta^d\Big(p^{(1)}+p^{(2)}\Big)e^{-\sum_{n\geq1}(-1)^n\hat{a}_n^{\dagger(1)}\hat{a}_n^{\dagger(2)}}|0,p^{(1)}\rangle|0,p^{(2)}\rangle\,.
\end{aligned} \tag{A.55}$$

In particular we have the bra-ket correspondence for eigenstates of x_n:

$${}_{12}\langle V_2|x_0,x_n\rangle_2 = {}_1\langle x_0,(-1)^nx_n|\,. \tag{A.56}$$

- Compute the n-string vertex for coherent states in terms of unknown Neumann coefficients $V^{rs}_{(n)}, V^{rs}_{0(n)}, V^{rs}_{00(n)}$

$$\begin{aligned}
|V_n\rangle &= \int\frac{d^dp^{(1)}}{(2\pi)^d}\cdots\frac{d^dp^{(n)}}{(2\pi)^d}(2\pi)^d\delta^d\left(p^{(1)}+\cdots+p^{(n)}\right)\\
&\quad\times e^{-\frac{1}{2}\hat{a}^{\dagger(r)}V^{rs}_{(n)}\hat{a}^{\dagger(s)}-p^{(r)}V^{rs}_{0(n)}\hat{a}^{\dagger(s)}-\frac{1}{2}p^{(r)}V^{rs}_{00(n)}p^{(s)}}|p^{(i)}\rangle\,,
\end{aligned} \tag{A.57}$$

$$\begin{aligned}
\langle\Psi_1|\cdots\langle\Psi_n|V_n\rangle &= (2\pi)^d\delta^d\left(p^{(1)}+\cdots+p^{(n)}\right)\\
&\quad\times e^{-\frac{1}{2}\mu^{(r)*}V^{rs}_{(n)}\mu^{(s)*}-p^{(r)}V^{rs}_{0(n)}\mu^{(s)*}-\frac{1}{2}p^{(r)}V^{rs}_{00(n)}p^{(s)}}\,.
\end{aligned} \tag{A.58}$$

- Compute the trace of the Moyal images of n coherent states in MSFT

$$\begin{aligned}
&\int d^d\bar{x}\,\mathrm{Tr}\left(\tilde{A}_1(\bar{x},\xi)\star\tilde{A}_2(\bar{x},\xi)\star\cdots\star\tilde{A}_n(\bar{x},\xi)\right) =\\
&\quad= (-1)^{\frac{Nd}{2}}\left(\det((1+m_0)^n-(1-m_0)^n)\right)^{-\frac{d}{2}}2^{nNd}(1+\bar{w}w)^{-\frac{nd}{8}}\\
&\quad\times(2\pi)^d\delta^d\left(p^{(1)}+\cdots+p^{(n)}\right)e^{E^{(n)}}\,,
\end{aligned} \tag{A.59}$$

$$\begin{aligned}
E^{(n)} &= -\frac{1}{2}\sum_{r,s}\mu^{(r)*}C\left(2K^{*-1}m_0\mathcal{O}_{(s-r)}(m_0)K^*-\delta_{r,s}\right)\mu^{(s)*} -\\
&\quad -2\sum_{r,s}p^{(r)}\bar{W}K^{*-1}m_0\mathcal{O}_{(s-r)}(m_0)K^*\mu^{(s)*} -
\end{aligned}$$

[23] Here we introduce the *bra* coherent state. This is a different convention from that in [9].

$$-\frac{1}{2}\sum_{r,s} p^{(r)} \left(2\bar{W}K^{*-1}m_0\mathcal{O}_{(s-r)}(m_0)K^*W\right) p^{(s)}\,,$$

$$m_0 \,:=\, M_0\sigma = \begin{pmatrix} 0 & \frac{i\theta}{2l_s^2}\kappa_e \\ -\frac{2il_s^2}{\theta}T\kappa_o^{-1}\bar{T} & 0 \end{pmatrix} \tag{A.60}$$

where we used

$$\begin{aligned} CK^{*-1}m_0 &= -\bar{K}^*\sigma\,, \\ CK^{*-1}m_0K^* &= -K^{*-1}m_0K^*C\,, \\ \mathcal{O}_{(s-r)}(m_0) &= -\mathcal{O}_{(r-s)}(-m_0)\,. \end{aligned} \tag{A.61}$$

We note

$$\tilde{m}_0^* := K^{*-1}m_0K^* = \begin{pmatrix} 0 & -\kappa_e^{\frac{1}{2}}T\kappa_o^{-\frac{1}{2}} \\ -\kappa_o^{-\frac{1}{2}}\bar{T}\kappa_e^{\frac{1}{2}} & 0 \end{pmatrix} = -\tilde{m}_0\,. \tag{A.62}$$

The sign is changed compared to that in [9] because we used *bra* coherent state $\langle\Psi_c|$ to define the Moyal field $\tilde{A}$.

- The Neumann coefficients in the matter sector are obtained by identifying the Fock space and MSFT expressions and comparing the exponents[24]

$$\int d^d\bar{x}\,\mathrm{Tr}\left(\tilde{A}_1(\bar{x},\xi)\star\tilde{A}_2(\bar{x},\xi)\star\cdots\star\tilde{A}_n(\bar{x},\xi)\right) = \rho\langle\Psi_1|\langle\Psi_2|\cdots\langle\Psi_n|V_n\rangle\,, \tag{A.63}$$

$$\rho = (-1)^{\frac{Nd}{2}}\left(\det((1+m_0)^n-(1-m_0)^n)\right)^{-\frac{d}{2}}2^{nNd}(1+\bar{w}w)^{-\frac{nd}{8}} \tag{A.64}$$

and using momentum conservation $\delta^d(p^{(1)}+p^{(2)}+\cdots+p^{(n)})$. Then one has the Neumann coefficients

$$\begin{aligned} V^{rs}_{(n)} &= C\left(2K^{*-1}m_0\mathcal{O}_{(s-r)}(m_0)K^*-\delta_{r,s}\right)\,, \\ V^{rs}_{0(n)} &= -2K^{*-1}m_0\mathcal{O}_{(s-r)}(-m_0)K^*W-\frac{2}{n}W\,, \\ V^{rs}_{00(n)} &= 2\bar{W}K^{*-1}m_0\mathcal{O}_{(s-r)}(m_0)K^*W-\frac{2}{n}\bar{W}W\,. \end{aligned} \tag{A.65}$$

They satisfy Neumann matrix algebra as in ref. [9]. For the 3-string vertex we write them explicitly

$$\begin{aligned} \mathcal{M}^{(0)} &:= CV^{rr}_{(3)} = \frac{\tilde{m}_0^{*2}-1}{\tilde{m}_0^{*2}+3}\,, \qquad \mathcal{M}^{(\pm)} := CV^{r,r\pm1}_{(3)} = \frac{2\pm\tilde{m}_0^*}{\tilde{m}_0^{*2}+3}\,, \\ \mathcal{V}^{(0)} &:= V^{rr}_{0(3)} = \frac{4\tilde{m}_0^{*2}}{3(\tilde{m}_0^{*2}+3)}\kappa_e^{-\frac{1}{2}}\frac{wl_s}{\sqrt{2}}\,, \\ \mathcal{V}^{(\pm)} &:= V^{r,r\pm1}_{0(3)} = \frac{-2\tilde{m}_0^{*2}\mp6\tilde{m}_0^*}{3(\tilde{m}_0^{*2}+3)}\kappa_e^{-\frac{1}{2}}\frac{wl_s}{\sqrt{2}}\,, \end{aligned}$$

[24]Here we defined the Witten's $*$ product using the *ket* $|V_3\rangle$ as eq. (2.8) which is different convention from that in [9]. Also, here we have included the overall normalization ρ which does not play a role in the computation of the Neumann coefficients.

$$V_{00} : = V^{rr}_{00(3)} = l_s^2\, \bar{w}\kappa_e^{-\frac{1}{2}} \frac{t\bar{t}}{t\bar{t}+3}\kappa_e^{-\frac{1}{2}} w \tag{A.66}$$

where we redefined as $V_{00}^{r,s} = V_{00}\delta_{r,s}$ using momentum conservation. We used the notation $\bar{w} = (w_e, 0)$, and $t = \kappa_e^{1/2} T \kappa_o^{-1/2}$.

- Fermionic ghost Neumann coefficients for the 3-vertex can be derived from matter Neumann coefficients for the 6-vertex

$$\begin{aligned} X^{rs} &: = (-1)^{r+s}\sqrt{\kappa_n}(V^{r,s}_{(6)} - V^{r,s+3}_{(6)})\frac{1}{\sqrt{\kappa_n}}\,, \\ X^{r\,s}_{\ \,0} &: = (-1)^{r+s}\sqrt{\kappa_n}(V^{r,s}_{\ \ 0(6)} - V^{r,s+3}_{\ \ 0(6)}) l_s^{-1}\,. \end{aligned} \tag{A.67}$$

This gives

$$\begin{aligned} X^{(0)} &= C\frac{\hat{m}_0^{*2}-1}{3\hat{m}_0^{*2}+1}\,, \quad X^{(+)} = -C\frac{2\hat{m}_0^{*}+2\hat{m}_0^{*2}}{3\hat{m}_0^{*2}+1}\,, \quad X^{(-)} = C\frac{2\hat{m}_0^{*}-2\hat{m}_0^{*2}}{3\hat{m}_0^{*2}+1}\,, \\ X^{(0)}_{\ \ 0} &= \frac{4\hat{m}_0^{*2}}{3\hat{m}_0^{*2}+1}\frac{w}{\sqrt{2}}\,, \quad X^{(+)}_{\ \ 0} = \frac{2\hat{m}_0^{*}-2\hat{m}_0^{*2}}{3\hat{m}_0^{*2}+1}\frac{w}{\sqrt{2}}\,, \quad X^{(-)}_{\ \ 0} = -\frac{2\hat{m}_0^{*}+2\hat{m}_0^{*2}}{3\hat{m}_0^{*2}+1}\frac{w}{\sqrt{2}}\,, \end{aligned} \tag{A.68}$$

where we defined

$$\hat{m}_0^{*} := \sqrt{\kappa_n}\tilde{m}_0^{*}\frac{1}{\sqrt{\kappa_n}} = \sqrt{\kappa_n}K^{*-1}m_0 K^{*}\frac{1}{\sqrt{\kappa_n}}\,. \tag{A.69}$$

B. Derivation of regularized matrix formula

Here we sketch a derivation of fundamental formulas for regularized matrices presented in section 2.1.3. We begin from the defining relations in eq. (2.37). The first two equations imply eq. (2.43) and then from the remaining equations we have

$$\sum_o Q_{eo}(v'_o)^2 = (\kappa'_e)^{-1}\,, \qquad \sum_e Q_{eo}(w'_e)^2 = (\kappa'_o)^{-1}\,, \tag{B.1}$$

where

$$Q_{eo} := \frac{1}{\kappa'_e - \kappa'_o}\,, \tag{B.2}$$

with $e = \pm 2, \pm 4, \ldots \pm 2N$, and $o = \pm 1, \pm 3, \ldots \pm (2N-1)$. Now we regard $Q = (Q_{eo})$ as a $2N \times 2N$ matrix and compute its inverse

$$(Q^{-1})_{oe} = (\kappa'_e - \kappa'_o)\frac{\prod_{o' \neq o}(\kappa'_e - \kappa'_{o'})\prod_{e' \neq e}(\kappa'_o - \kappa'_{e'})}{\prod_{e' \neq e}(\kappa'_e - \kappa'_{e'})\prod_{o' \neq o}(\kappa'_o - \kappa'_{o'})}\,. \tag{B.3}$$

To prove the above formula, it is convenient to define a rational function $f(z)$ which is determined by the setup $(N, \kappa'_e, \kappa'_o)$ uniquely:

$$f(z) := \frac{\prod_{o'}(z - \kappa'_{o'})}{\prod_{e'}(z - \kappa'_{e'})}\,. \tag{B.4}$$

JHEP07(2003)027

Next we compute $(Q^{-1}Q)_{oo''}$. We use contour integration and residues, where we denote the residue of $f(z)$ at $z = z_0$ as $\mathrm{Res}_{z=z_0} f(z)$ and assume that the frequencies κ'_e, κ'_o are nondegenerate and finite

$$\begin{aligned}
&\sum_e \frac{\kappa'_e - \kappa'_o}{\kappa'_e - \kappa'_{o''}} \frac{\prod_{o'\neq o}(\kappa'_e - \kappa'_{o'}) \prod_{e'\neq e}(\kappa'_o - \kappa'_{e'})}{\prod_{e'\neq e}(\kappa'_e - \kappa'_{e'}) \prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} = \\
&\quad = \sum_e \frac{-\mathrm{Res}_{z=\kappa'_e} f(z)}{(\kappa'_e - \kappa'_o)(\kappa'_e - \kappa'_{o''})} \frac{\prod_{e'}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \\
&\quad = -\frac{\prod_{e'}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \sum_e \mathrm{Res}_{z=\kappa'_e} \frac{f(z)}{(z-\kappa'_o)(z-\kappa'_{o''})} \\
&\quad = \frac{\prod_{e'}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \oint_{z=\kappa'_o,\kappa'_{o''}} \frac{dz}{2\pi i} \frac{f(z)}{(z-\kappa'_o)(z-\kappa'_{o''})} \\
&\quad = \frac{\prod_{e'}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \mathrm{Res}_{z=\kappa'_o} \frac{f(z)}{(z-\kappa'_o)^2} \delta_{o,o''} = \delta_{o,o''} \,.
\end{aligned} \tag{B.5}$$

This shows that we have the correct inverse matrix Q^{-1}.

Similarly, we can obtain $(v'_o)^2, (w'_e)^2$ eqs. (2.45), (2.44) as follows:[25]

$$\begin{aligned}
(v'_o)^2 &= \sum_e (Q^{-1})_{oe} (\kappa'_e)^{-1} = -\frac{\prod_{e'}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \sum_e \frac{1}{\kappa'_e} \frac{\prod_{o'\neq o}(\kappa'_e - \kappa'_{o'})}{\prod_{e'\neq e}(\kappa'_e - \kappa'_{e'})} \\
&= -\frac{\prod_{e'}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \sum_e \mathrm{Res}_{z=\kappa'_e} \frac{f(z)}{z(z-\kappa'_o)} = \frac{\prod_{e'}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \mathrm{Res}_{z=0} \frac{f(z)}{z(z-\kappa'_o)} \\
&= \frac{\prod_{o'\neq o}\kappa'_{o'} \prod_{e'}(\kappa'_{e'} - \kappa'_o)}{\prod_{e'}\kappa'_{e'} \prod_{o'\neq o}(\kappa'_{o'} - \kappa'_o)} = \frac{\prod_{o'>0,o'\neq|o|}\kappa^2_{o'} \prod_{e'>0}(\kappa^2_{e'} - \kappa^2_{|o|})}{2\prod_{e'>0}\kappa^2_{e'} \prod_{o'>0,o'\neq|o|}(\kappa^2_{o'} - \kappa^2_{|o|})} \,,
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
(w'_e)^2 &= \sum_o (Q^{-1})_{oe} (\kappa'_o)^{-1} = \frac{\prod_{o'}(\kappa'_e - \kappa'_{o'})}{\prod_{e'\neq e}(\kappa'_e - \kappa'_{e'})} \sum_o \frac{1}{\kappa'_o} \frac{\prod_{e'\neq e}(\kappa'_o - \kappa'_{e'})}{\prod_{o'\neq o}(\kappa'_o - \kappa'_{o'})} \\
&= \frac{\prod_{o'}(\kappa'_e - \kappa'_{o'})}{\prod_{e'\neq e}(\kappa'_e - \kappa'_{e'})} \sum_o \mathrm{Res}_{z=\kappa'_o} \frac{f(z)}{z(z-\kappa'_e)} = -\frac{\prod_{o'}(\kappa'_e - \kappa'_{o'})}{\prod_{e'\neq e}(\kappa'_e - \kappa'_{e'})} \mathrm{Res}_{z=0} \frac{f(z)}{z(z-\kappa'_e)} \\
&= \frac{\prod_{e'\neq e}\kappa'_{e'} \prod_{o'}(\kappa'_e - \kappa'_{o'})}{\prod_{o'}\kappa'_{o'} \prod_{e'\neq e}(\kappa'_e - \kappa'_{e'})} = \frac{\prod_{e'>0,e'\neq|e|}\kappa^2_{e'} \prod_{o'>0}(\kappa^2_{|e|} - \kappa^2_{o'})}{2\prod_{o'>0}\kappa^2_{o'} \prod_{e'>0,e'\neq|e|}(\kappa^2_{|e|} - \kappa^2_{e'})}
\end{aligned} \tag{B.7}$$

where we assumed κ'_e, κ'_o are nonzero and eqs. (2.42). We note that the above formula (B.6), (B.7) can also be rewritten as

$$(v'_o)^2 = \frac{1}{\kappa'_o} \mathrm{Res}_{z=\kappa'_o} \frac{f(0)}{f(z)} \,, \qquad (w'_e)^2 = \frac{1}{\kappa'_e} \mathrm{Res}_{z=\kappa'_e} \frac{f(z)}{f(0)} \,. \tag{B.8}$$

Now we consider the open string limit (A.19). By setting the open string limit $\kappa'_e = e, \kappa'_o = o, N \to \infty$, the rational function $f(z)$ (B.4) becomes

$$\frac{f(z)}{f(0)} = \frac{\prod_{n=1}^{\infty}(1 - \frac{z^2}{(2n-1)^2})}{\prod_{n=1}^{\infty}(1 - \frac{z^2}{4n^2})} = \left(\cos\frac{\pi z}{2}\right)\left(\frac{2}{\pi z}\sin\frac{\pi z}{2}\right)^{-1} = \frac{\pi z}{2\tan\frac{\pi z}{2}} \,. \tag{B.9}$$

[25] We choose the sign convention of w'_e, v'_o such that they are consistent with eq. (2.35).

With this formula and eq. (B.8) we can show that the regularized quantities reduce to the original ones in eq. (2.35).

C. bc-ghost sector in position space

Here we consider the ghost position space representation of the n-point string vertices for $n = 1, 2, 3$, starting with the Fock space formalism.

Identity state.

$$|I\rangle = \left(\sum_{o>0}(-1)^{\frac{o-1}{2}}\hat{b}_{-o}\right)\left(\hat{b}_0 + 2\sum_{e>0}(-1)^{\frac{e}{2}}\hat{b}_{-e}\right) e^{\sum_{n=1}^{\infty}(-1)^n \hat{c}_{-n}\hat{b}_{-n}}\hat{c}_0\hat{c}_1|\Omega\rangle, \tag{C.1}$$

$$\langle c_0, x_n, y_n|I\rangle = \frac{i}{\sqrt{2}}\left(\sum_{o>0}(-1)^{\frac{o-1}{2}}x_o\right)\prod_{e>0}\left((i\sqrt{2}x_e)(\sqrt{2}y_e - 2(-1)^{\frac{e}{2}}c_0)\right). \tag{C.2}$$

This is BRST invariant [2] although the form is rather complicated.

Reflector.

$${}_{12}\langle V_2| = {}_1\langle\Omega|\hat{c}^{(1)}_{-1}\,{}_2\langle\Omega|\hat{c}^{(2)}_{-1}e^{-\sum_{n=1}^{\infty}(-1)^n(\hat{c}^{(1)}_n\hat{b}^{(2)}_n + \hat{c}^{(2)}_n\hat{b}^{(1)}_n)}(\hat{c}^{(1)}_0 + \hat{c}^{(2)}_0)\,, \tag{C.3}$$

$$|V_2\rangle_{12} = (\hat{b}^{(1)}_0 - \hat{b}^{(2)}_0)e^{\sum_{n=1}^{\infty}(-1)^n(\hat{c}^{(1)}_{-n}\hat{b}^{(2)}_{-n} + \hat{c}^{(2)}_{-n}\hat{b}^{(1)}_{-n})}\hat{c}^{(1)}_0\hat{c}^{(1)}_1|\Omega\rangle_1\,\hat{c}^{(2)}_0\hat{c}^{(2)}_1|\Omega\rangle_2\,, \tag{C.4}$$

$$\begin{aligned}{}_1\langle c^{(1)}_0, x^{(1)}_n, y^{(1)}_n|{}_2\langle c^{(2)}_0, x^{(2)}_n, y^{(2)}_n|V_2\rangle_{12} &= \left(c^{(1)}_0 + c^{(2)}_0\right)\times \\ &\times\prod_{n=1}^{\infty}\left(-2i((-1)^n x^{(1)}_n + x^{(2)}_n)\left((-1)^n y^{(1)}_n + y^{(2)}_n\right)\right) \\ &= \langle c^{(1)}_0, x^{(1)}_n, y^{(1)}_n| - c^{(2)}_0, -(-1)^n x^{(2)}_n, -(-1)^n y^{(2)}_n\rangle\,.\end{aligned} \tag{C.5}$$

3-string vertex.

$$|V_3\rangle_{123} = e^{\sum_{r,s=1}^{3}\left(-\hat{c}^{\dagger(r)}X^{rs}\hat{b}^{\dagger(s)} - \hat{c}^{\dagger(r)}X^{rs}_{\ 0}\hat{b}^{(s)}_0\right)}\hat{c}^{(1)}_0\hat{c}^{(1)}_1|\Omega\rangle_1\hat{c}^{(2)}_0\hat{c}^{(2)}_1|\Omega\rangle_2\hat{c}^{(3)}_0\hat{c}^{(3)}_1|\Omega\rangle_3\,, \tag{C.6}$$

$$\begin{aligned}&{}_1\langle c^{(1)}_0, x^{(1)}_n, y^{(1)}_n|{}_2\langle c^{(2)}_0, x^{(2)}_n, y^{(2)}_n|{}_3\langle c^{(3)}_0, x^{(3)}_n, y^{(3)}_n|V_3\rangle_{123} = \\ &= -\det_{r,s,n,m}(\delta^{r,s}\delta_{n,m} + X^{rs}_{nm})\left(c^{(1)}_0 - \bar{w}y^{(1)}_e\right)\left(c^{(2)}_0 - \bar{w}y^{(2)}_e\right)\times \\ &\times\left(c^{(3)}_0 - \bar{w}y^{(3)}_e\right)e^{i\sum_{r,s}y^{(r)}\left(\frac{1-X}{1+X}\right)^{rs}x^{(s)}}\end{aligned} \tag{C.7}$$

where we used the relation [21, 22, 23],

$$X^{rs}_{\ 0} = (\delta^{rs} + X^{rs})\frac{w}{\sqrt{2}}\,. \tag{C.8}$$

Witten's star product in ghost position space

$$\begin{aligned}\Psi_1 \star^W \Psi_2(c_0, x_n, y_n) &= \int dc^{(2)}_0 dc^{(3)}_0\frac{dx^{(2)}_n dy^{(2)}_n}{-2i}\frac{dx^{(3)}_n dy^{(3)}_n}{-2i}\times \\ &\times{}_1\langle c_0, x_n, y_n|{}_2\langle c^{(2)}_0, \widetilde{x^{(2)}_n}, y^{(2)}_n|{}_3\langle c^{(3)}_0, \widetilde{x^{(3)}_n}, y^{(3)}_n|V_3\rangle_{123}\times\end{aligned}$$

$$
\begin{aligned}
&\times\Psi_1\left(c_0^{(2)}, x_n^{(2)}, y_n^{(2)}\right)\Psi_1\left(c_0^{(3)}, x_n^{(3)}, y_n^{(3)}\right) \\
&= \int dc_0^{(2)} dc_0^{(3)} \frac{dx_n^{(2)} dy_n^{(2)}}{-2i}\frac{dx_n^{(3)} dy_n^{(3)}}{-2i} \times \\
&\quad \times {}_1\langle c_0, x_n, y_n|{}_2\langle c_0^{(2)}, x_n^{(2)}, y_n^{(2)}|{}_3\langle c_0^{(3)}, x_n^{(3)}, y_n^{(3)}|V_3\rangle_{123} \times \\
&\quad \times\Psi_1\left(-c_0^{(2)}, -(-1)^n x_n^{(2)}, -(-1)^n y_n^{(2)}\right) \times \\
&\quad \times\Psi_2\left(-c_0^{(3)}, -(-1)^n x_n^{(3)}, -(-1)^n y_n^{(3)}\right)
\end{aligned} \tag{C.9}
$$

where

$$
{}_1\langle \widetilde{c_0, x_n, y_n}| := {}_{12}\langle V_2|c_0, x_n, y_n\rangle_2 = {}_1\langle -c_0, -(-1)^n x_n, -(-1)^n y_n| \,. \tag{C.10}
$$

D. Moyal $\star$ for $(DD)_b$ $(NN)_c$ split strings

In this appendix, we examine the remaining choice of the midpoint boundary condition for the split string variables as compared with our discussion in section 2.3. Namely, we consider Dirichlet boundary condition for $b(\sigma)$ and Neumann for $c(\sigma)$ at the midpoint $\sigma = \pi/2$. In this case the left and right half of $b(\sigma)$: $l^b(\sigma), r^b(\sigma)$ satisfy Dirichlet boundary condition at both $\sigma = 0, \pi/2$, and those of $c(\sigma)$: $l^c(\sigma), r^b(\sigma)$ satisfy Neumann at $\sigma = 0, \pi/2$. The $l^b(\sigma), r^b(\sigma)$ and $l^c(\sigma), r^c(\sigma)$ are expanded in terms of *even* sine/cosine modes respectively

$$
l^b(\sigma) = \frac{2}{\pi}\sigma\bar{b} + i\sqrt{2}\sum_{e=2}^{\infty} l_e^b \sin e\sigma \,, \qquad r^b(\sigma) = \frac{2}{\pi}\sigma\bar{b} + i\sqrt{2}\sum_{e=2}^{\infty} r_e^b \sin e\sigma \,, \tag{D.1}
$$

$$
l^c(\sigma) = \bar{c} + \sqrt{2}\sum_{e=2}^{\infty} l_e^c(\cos e\sigma - i^e) \,, \qquad r^c(\sigma) = \bar{c} + \sqrt{2}\sum_{e=2}^{\infty} r_e^c(\cos e\sigma - i^e) \,. \tag{D.2}
$$

From eqs. (2.69) (2.70) (2.27) (A.12), we have relations between split and full string variables:

$$
\bar{b} = \bar{\tilde{w}} x_o \,, \qquad l_e^b = x_e + \tilde{T} x_o \,, \qquad r_e^b = -x_e + \tilde{T} x_o \,, \tag{D.3}
$$

$$
\bar{c} = c_0 - \bar{w} y_e \,, \qquad l_e^c = y_e + T y_o \,. \qquad r_e^c = y_e - T y_o \,. \tag{D.4}
$$

With this setup Witten type product in the split string formulation becomes:

$$
\tilde{A}' * \tilde{B}'(\bar{b}, \bar{c}, l_e^b, l_e^c, r_e^b, r_e^c) = \int \prod_{e>0}\left(id\eta_e^b d\eta_e^c\right) \tilde{A}'(\bar{b}, \bar{c}, l_e^b, l_e^c, \eta_e^b, \eta_e^c)\tilde{B}'(\bar{b}, \bar{c}, \eta_e^b, -\eta_e^c, r_e^b, r_e^c) \,. \tag{D.5}
$$

where the split string and full string fields in position space are the same

$$
\tilde{A}'(\bar{b}, \bar{c}, l_e^b, l_e^c, r_e^b, r_e^c) \sim \Psi(c_0, x_n, y_n) \tag{D.6}
$$

by substituting on the right hand side the inverse maps obtained in eqs.(D.3) (D.4)

$$
x_e = \frac{1}{2}(l_e^b - r_e^b) \,, \quad x_o = \tilde{u}_o \bar{b} + \frac{1}{2}\bar{S}_{oe}(l_e^b + r_e^b) \,, \tag{D.7}
$$

$$
c_0 = \bar{c} + \frac{1}{2}\bar{w}_e(l_e^c + r_e^c) \,, \quad y_e = \frac{1}{2}(l_e^c + r_e^c) \,, \quad y_o = \frac{1}{2}R_{oe}(l_e^c - r_e^c) \,. \tag{D.8}
$$

These relations are valid only when $\bar{w}w = \infty$ in the limit: $\kappa_e = e, \kappa_o = o, N = \infty$.

As the next step, we consider the Moyal formulation, including the regularization with (N, κ_e, κ_o). Comparing eqs.(D.3) (D.4) (D.5) with eq.(2.63), we identify

$$x \sim \tilde{T} x_o\,, \qquad y \sim 2x_e\,, \qquad x' \sim -T y_o\,, \qquad y' \sim 2y_e\,,$$

and *define* new variables with even index E by

$$x_E := \tilde{T} x_o\,, \quad y_E := T y_o\,. \tag{D.9}$$

At the limit $\kappa_e = e, \kappa_o = o, N = \infty$, $\tilde{T}$ has a zero mode $\tilde{u}$ (2.34) and we meet as usual the associativity anomaly, but at finite N everything is well-defined. From eq. (2.63) (D.6) we obtain the Moyal image $A'(\bar{b}, \bar{c}, x_E, p_E, y_E, q_E)$ of the position space field $\Psi(c_0, x_n, y_n)$:

$$\begin{aligned}
&A'(\bar{b}, \bar{c}, x_E, p_E, y_E, q_E) = \\
&\quad = 2^{-2N} \int \prod_{e>0} \left(i^{-1} dx_e dy_e\right) e^{-2p_E x_e - 2q_E y_e} \tilde{A}'(\bar{b}, \bar{c}, x_e + x_E, y_e + y_E, -x_e + x_E, y_e - y_E) \\
&\quad = 2^{-2N} \int \prod_{e>0} \left(i^{-1} dx_e dy_e\right) e^{-2p_E x_e - 2q_E y_e} \Psi(\bar{c} + \bar{w} y_e, x_e, \tilde{u}\bar{b} + \bar{S} x_E, y_e, R y_E)\,,
\end{aligned} \tag{D.10}$$

and the corresponding Moyal $\star$ product becomes

$$\star = e^{-\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x_E}\frac{\overrightarrow{\partial}}{\partial p_E} + \frac{\overleftarrow{\partial}}{\partial y_E}\frac{\overrightarrow{\partial}}{\partial q_E} + \frac{\overleftarrow{\partial}}{\partial p_E}\frac{\overrightarrow{\partial}}{\partial x_E} + \frac{\overleftarrow{\partial}}{\partial q_E}\frac{\overrightarrow{\partial}}{\partial y_E}\right)}\,. \tag{D.11}$$

In this case, the above formula is more complicated than our previous choice due to the additional b-ghost midpoint mode $\bar{b}$.

E. Ghost butterfly projector with even modes

The butterfly projector in eq. (2.126) is based on the odd mode oscillators in eq. (2.10). There is another choice using even mode oscillators eq. (2.140)

$$\beta^b_e \star \hat{A}'_B = \beta^c_e \star \hat{A}'_B = \hat{A}'_B \star \beta^b_{-e} = \hat{A}'_B \star \beta^c_{-e} = 0\,, \quad \forall e > 0\,. \tag{E.1}$$

Explicitly, we have the even butterfly state

$$\hat{A}'_B = \xi_0\, 2^{-2N} \exp\left(-\sum_{e>0}\left(i x^b_e \kappa_e x^c_e + \frac{4i}{\theta'^2} p^b_e \kappa_e^{-1} p^c_e\right)\right) \tag{E.2}$$

in the Siegel gauge.

As we will see in the following, the even butterfly is the one defined by Gaiotto-Rastelli-Sen-Zwiebach(GRSZ) using twisted ghosts [25]. The conditions in eq. (E.1) correspond to

$$\begin{aligned}
\hat{b}_e|\Psi_B\rangle &= 0\,, & \hat{c}_e|\Psi_B\rangle &= 0\,, \\
\sum_{o>0}\left(\hat{b}_o U^{-1}_{-o,e} + \hat{b}_{-o} U^{-1}_{o,e}\right)|\Psi_B\rangle &= 0\,, & \sum_{o>0}\left(U_{e,-o}\hat{c}_o + U_{e,o}\hat{c}_{-o}\right)|\Psi_B\rangle &= 0
\end{aligned} \tag{E.3}$$

for all $e > 0$ in ordinary oscillator language. If we take a gaussian ansatz

$$|\Psi_B\rangle = \mathcal{N} \exp\left(\sum_{n,m\geq 1} \hat{c}_{-m} \tilde{V}^B_{mn} \hat{b}_{-n}\right) \hat{c}_1 |\Omega\rangle\,, \tag{E.4}$$

the above conditions become

$$\tilde{V}^B_{eo} = \tilde{V}^B_{oe} = \tilde{V}^B_{ee'} = 0\,, \qquad e, e', o > 0\,, \tag{E.5}$$

$$\sum_{o'>0} \tilde{V}^B_{o'o} U^{-1}_{-o',e} = -U^{-1}_{o,e}\,, \qquad e > 0\,, \tag{E.6}$$

$$\sum_{o'>0} \tilde{V}^B_{oo'} U_{e,-o'} = U_{e,o}\,, \qquad e > 0\,. \tag{E.7}$$

We can solve eq. (E.6) by comparing it with the matter one (A.44) as

$$\begin{aligned} \tilde{V}^B_{mn} &= -\frac{1}{\sqrt{n}} V^B_{nm} \sqrt{m} = -\sqrt{m} V^B_{mn} \frac{1}{\sqrt{n}} \\ &= \begin{cases} (-1)^{\frac{m+n}{2}} \frac{m}{m+n} \frac{\Gamma[\frac{m}{2}]\Gamma[\frac{n}{2}]}{\pi\Gamma[\frac{m+1}{2}]\Gamma[\frac{n+1}{2}]} & \text{for } m \text{ and } n \text{ odd} \\ 0 & \text{for } m \text{ or } n \text{ even}\,. \end{cases} \end{aligned} \tag{E.8}$$

We can check that this $\tilde{V}^B$ satisfies eq. (E.7). The ghost butterfly which we have obtained in MSFT as above can be identified with GRSZ's (twisted) ones [25]:[26]

$$\hat{A}'_B \quad \leftrightarrow \quad |\Psi_B\rangle = \exp\left(-\frac{1}{2} L'_{-2}\right) |\Omega'\rangle \quad \text{for} \quad \kappa_e = e, \kappa_o = o, N = \infty\,.$$

The relation between the matter and (twisted) ghost generating functions is obtained by using [28]

$$\frac{\partial}{\partial z} \tilde{S}(w,z) = S(z,w)$$

where we defined the generating functions

$$S(z,w) := \sum_{m,n=1}^{\infty} \sqrt{mn} (-z)^{m-1} (-w)^{n-1} S_{mn} = \frac{1}{(z-w)^2} - \frac{f'(z) f'(w)}{(f(z)-f(w))^2}\,, \tag{E.9}$$

$$\tilde{S}(z,w) := \sum_{m,n=1}^{\infty} (-z)^{m-1} (-w)^n \tilde{S}_{mn} = -\frac{w}{z(w-z)} + \frac{f(w)}{f(z)} \frac{f'(z)}{f(w)-f(z)}\,, \tag{E.10}$$

$$|S\rangle = \exp\left(-\frac{1}{2} \sum_{m,n=1}^{\infty} \hat{a}^\dagger_m S_{mn} \hat{a}^\dagger_n\right) |\Omega\rangle\,, \quad |\tilde{S}\rangle = \exp\left(\sum_{m,n=1}^{\infty} \hat{c}_{-m} \tilde{S}_{mn} \hat{b}_{-n}\right) |\Omega'\rangle \tag{E.11}$$

namely

$$\tilde{S}_{mn} = -\frac{1}{\sqrt{n}} S_{nm} \sqrt{m} = -\sqrt{m} S_{mn} \frac{1}{\sqrt{n}}\,. \tag{E.12}$$

Here we have $V^B_{mn} = S_{mn}$ for the conformal mapping $f(z) = \frac{z}{\sqrt{1+z^2}}$ which represents (canonical) butterfly state $e^{-\frac{1}{2}L_{-2}}|\Omega\rangle$ [27] and the relation eq. (E.8).

[26]There is a correspondence for the vacuum: $\hat{c}_1|\Omega\rangle \sim |\Omega'\rangle$.

F. Algebra of gaussian operators

We discuss algebraic relations for gaussians constructed from bosonic and fermionic oscillators which we used to compute the propagator (4.17) in MSFT.

For bosonic oscillators: $a, a^\dagger, [a, a^\dagger] = 1$, we can prove a formula

$$e^{a^\dagger A a^\dagger + aBa} = e^{-\frac{1}{2}\mathrm{Tr}\log(\cos(2\sqrt{AB}))} e^{\frac{1}{2}a^\dagger \tan\left(2\sqrt{AB}\right)\sqrt{AB}B^{-1}a^\dagger} e^{-a^\dagger \log\left(\cos\left(2\sqrt{AB}\right)\right)a} e^{\frac{1}{2}aB\frac{\tan\left(2\sqrt{AB}\right)}{\sqrt{AB}}a} \tag{F.1}$$

using a similar method to appendix A in [32]. Here A, B are symmetric matrices: $\bar{A} = A, \bar{B} = B$. Then we have

$$e^{\eta A\eta + \frac{\partial}{\partial\eta}B\frac{\partial}{\partial\eta}} e^{i\xi\eta} = e^{-\frac{1}{2}\mathrm{Tr}\log(\cos(2\sqrt{AB}))} e^{\frac{1}{2}\eta\tan\left(2\sqrt{AB}\right)\sqrt{AB}B^{-1}\eta - \frac{1}{2}\xi B\frac{\tan\left(2\sqrt{AB}\right)}{\sqrt{AB}}\xi + i\eta\frac{1}{\cos\left(2\sqrt{AB}\right)}\xi} \tag{F.2}$$

where we used the relation

$$\left[\frac{\partial}{\partial\eta}, \eta\right] = 1\,, \qquad e^{\eta C\frac{\partial}{\partial\eta}} e^{i\xi\eta} e^{-\eta C\frac{\partial}{\partial\eta}} = e^{\eta e^C \xi}\,. \tag{F.3}$$

We obtain the formula for the propagator

$$\int d^M\xi\, e^{-i\xi\eta} e^{\eta' A\eta' + \frac{\partial}{\partial\eta'}B\frac{\partial}{\partial\eta'}} e^{i\xi\eta'} = \tag{F.4}$$

$$= (2\pi)^{\frac{M}{2}}\, e^{-\frac{1}{2}\mathrm{Tr}\log\left(B\frac{\sin(2\sqrt{AB})}{\sqrt{AB}}\right)} e^{-\frac{1}{2}\eta\frac{\sqrt{AB}}{\tan(2\sqrt{AB})}B^{-1}\eta - \frac{1}{2}\eta'\frac{\sqrt{AB}}{\tan(2\sqrt{AB})}B^{-1}\eta' + \eta'\frac{\sqrt{AB}}{\sin(2\sqrt{AB})}B^{-1}\eta}\,.$$

When the momentum is nonzero in (4.17), we need the modified version of the above formula:

$$\int d^M\xi\, e^{-i\xi\eta} e^{\eta' A\eta' + \frac{\partial}{\partial\eta'}B\frac{\partial}{\partial\eta'} + C\eta'} e^{i\xi\eta'} =$$

$$= (2\pi)^{\frac{M}{2}} e^{-\frac{1}{2}\mathrm{Tr}\log\left(B\frac{\sin(2\sqrt{AB})}{\sqrt{AB}}\right)} e^{-\frac{1}{2}\eta\frac{\sqrt{AB}}{\tan(2\sqrt{AB})}B^{-1}\eta - \frac{1}{2}\eta'\frac{\sqrt{AB}}{\tan(2\sqrt{AB})}B^{-1}\eta' + \eta'\frac{\sqrt{AB}}{\sin(2\sqrt{AB})}B^{-1}\eta} \times$$
$$\times\, e^{-\frac{1}{4}CA^{-1}\left(1 - \frac{\tan\sqrt{AB}}{\sqrt{AB}}\right)C + \frac{1}{2}(\eta+\eta')\frac{\tan\sqrt{AB}}{\sqrt{AB}}C}\,. \tag{F.5}$$

For fermionic oscillators: $a, a^\dagger, \{a_i, a_j^\dagger\} = \delta_{ij}$, we have a similar formula

$$e^{\bar{a}^\dagger A a^\dagger + \bar{a}Ba} = e^{\frac{1}{2}\mathrm{Tr}(\log\cosh(2\sqrt{AB}))} e^{\frac{1}{2}\bar{a}^\dagger \tanh(2\sqrt{AB})\sqrt{AB}B^{-1}a^\dagger} \times$$
$$\times\, e^{-\bar{a}^\dagger \log(\cosh(2\sqrt{AB}))a} e^{\frac{1}{2}\bar{a}B\frac{\tanh(2\sqrt{AB})}{\sqrt{AB}}a}\,, \tag{F.6}$$

where A, B are antisymmetric matrices $\bar{A} = -A, \bar{B} = -B$. Noting

$$\left\{\eta, \frac{\partial}{\partial\eta}\right\} = 1\,, \qquad e^{\bar{\eta}C\frac{\partial}{\partial\eta}} e^{-\bar{\xi}\eta} e^{-\bar{\eta}C\frac{\partial}{\partial\eta}} = e^{\bar{\eta}e^C\xi}\,, \tag{F.7}$$

we obtain

$$e^{\bar{\eta}A\eta + \frac{\bar{\partial}}{\partial\eta}B\frac{\partial}{\partial\eta}} e^{-\bar{\xi}\eta} = e^{\frac{1}{2}\mathrm{Tr}(\log\cosh(2\sqrt{AB}))} e^{\frac{1}{2}\bar{\eta}\tanh(2\sqrt{AB})\sqrt{AB}B^{-1}\eta + \frac{1}{2}\bar{\xi}B\frac{\tanh 2\sqrt{AB}}{\sqrt{AB}}\xi + \bar{\eta}\frac{1}{\cosh 2\sqrt{AB}}\xi}\,.$$

By integration we have

$$\int d\xi\, e^{\bar{\xi}\eta} e^{\bar{\eta}' A\eta' + \frac{\bar{\partial}}{\partial\eta'}B\frac{\partial}{\partial\eta'}} e^{-\bar{\xi}\eta'} =$$

$$= \det{}^{\frac{1}{2}}\left(B\frac{\sinh 2\sqrt{AB}}{\sqrt{AB}}\right) e^{\frac{1}{2}\bar{\eta}\frac{\sqrt{AB}}{\tanh 2\sqrt{AB}}B^{-1}\eta + \frac{1}{2}\bar{\eta}'\frac{\sqrt{AB}}{\tanh 2\sqrt{AB}}B^{-1}\eta' - \bar{\eta}'\frac{\sqrt{AB}}{\sinh 2\sqrt{AB}}B^{-1}\eta}\,. \tag{F.8}$$

G. Neumann coefficients

G.1 Neumann coefficients from CFT

In this subsection, we give a short summary of the analytic expression of Neumann coefficients given in [2] which are obtained from conformal field theory. We introduce a set of numbers A_n, B_n $(n = 0, 1, 2, \ldots)$ which appear in the Taylor expansion,

$$\left(\frac{1+ix}{1-ix}\right)^{1/3} = \sum_{e\geq 0} A_e x^e + i\sum_{o>0} A_o x^o\,, \qquad \left(\frac{1+ix}{1-ix}\right)^{2/3} = \sum_{e\geq 0} B_e x^e + i\sum_{o>0} B_o x^o\,. \quad \text{(G.1)}$$

From these data, the Neumann coefficients ($N^{(0,\pm)}_{nm}$ for matter sector and $\tilde{N}^{(0,\pm)}_{nm}$ for ghost sector) are written as follows. First when $n, m > 0$ and $n \neq m$,

$$N^{(0)}_{nm} = \begin{cases} \frac{(-1)^n}{3}\left(\frac{A_nB_m+B_nA_m}{(n+m)} + \frac{A_nB_m-B_nA_m}{(n-m)}\right) & n+m = \text{even} \\ 0 & n+m = \text{odd}\,, \end{cases} \quad \text{(G.2)}$$

$$N^{(\pm)}_{nm} = \begin{cases} \frac{-(-1)^n}{6}\left(\frac{A_nB_m+B_nA_m}{(n+m)} + \frac{A_nB_m-B_nA_m}{(n-m)}\right) & n+m = \text{even} \\ \frac{\pm\sqrt{3}}{6}\left(\frac{A_nB_m-B_nA_m}{(n+m)} + \frac{A_nB_m+B_nA_m}{(n-m)}\right) & n+m = \text{odd}\,. \end{cases} \quad \text{(G.3)}$$

$$\tilde{N}^{(0)}_{nm} = \begin{cases} \frac{-(-1)^n}{3}\left(\frac{A_nB_m+B_nA_m}{(n+m)} - \frac{A_nB_m-B_nA_m}{(n-m)}\right) & n+m = \text{even} \\ 0 & n+m = \text{odd}\,, \end{cases} \quad \text{(G.4)}$$

$$\tilde{N}^{(\pm)}_{nm} = \begin{cases} \frac{(-1)^n}{6}\left(\frac{A_nB_m+B_nA_m}{(n+m)} - \frac{A_nB_m-B_nA_m}{(n-m)}\right) & n+m = \text{even} \\ \frac{\mp\sqrt{3}}{6}\left(\frac{A_nB_m-B_nA_m}{(n+m)} - \frac{A_nB_m+B_nA_m}{(n-m)}\right) & n+m = \text{odd}\,. \end{cases} \quad \text{(G.5)}$$

For the diagonal components $(n = m > 0)$, they are replaced by,

$$N^{(0)}_{nn} = \frac{1}{3n}\left(2(-1)^n\left(1+\sum_{k=1}^{n}(-1)^k A_k^2\right) - (-1)^n - A_n^2\right), \quad N^{(\pm)}_{nn} = -\frac{(-1)^n}{2n} - \frac{N^{(0)}_{nn}}{2}\,, \quad \text{(G.6)}$$

$$\tilde{N}^{(0)}_{nn} = N^{(0)}_{nn} - \frac{2(-1)^n A_n B_n}{3n}\,, \qquad \tilde{N}^{(\pm)}_{nn} = -\frac{(-1)^n}{2n} - \frac{1}{2}\tilde{N}^{(0)}_{nn}\,. \quad \text{(G.7)}$$

For the zero mode, we use

$$N^{(0)}_{0m} = \begin{cases} \frac{2}{3m}A_m & m = \text{even} \\ 0 & m = \text{odd}\,, \end{cases} \quad N^{(\pm)}_{0m} = \begin{cases} \frac{-1}{3m}A_m & m = \text{even} \\ \frac{\mp\sqrt{3}}{3m}A_m & m = \text{odd}\,, \end{cases} \quad N_{00} = -\frac{1}{2}\ln\frac{3^3}{4^2}\,, \quad \text{(G.8)}$$

$$\tilde{N}^{(0)}_{0m} = \begin{cases} \frac{-2}{3m}B_m & m = \text{even} \\ 0 & m = \text{odd}\,, \end{cases} \quad \tilde{N}^{(\pm)}_{0m} = \begin{cases} \frac{1}{3m}B_m & m = \text{even} \\ \frac{\mp\sqrt{3}}{3m}B_m & m = \text{odd}\,. \end{cases} \quad \text{(G.9)}$$

There are some differences in the convention to make direct comparison of these quantities with the corresponding ones obtained in Moyal language which are given in (A.66), (A.68). We summarize them as follows,

$$\mathcal{M}^{(0,\pm)}_{nm}(cft) \equiv -(-1)^n\sqrt{mn}N^{(0,\pm)}_{mn}\,,$$
$$\mathcal{V}^{(0,\pm)}_{n}(cft) \equiv -l_s\sqrt{n}N^{(0,\pm)}_{0n}\,,$$

$\frac{\mathcal{M}^{(0)}_{ee'}(5)}{\mathcal{M}^{(0)}_{ee'}(cft)}$	2	4	6	8	$\frac{\mathcal{M}^{(0)}_{ee'}(20)}{\mathcal{M}^{(0)}_{ee'}(cft)}$	2	4	6	8
2	1.15355	1.27359	1.43938	1.71214	2	1.02373	1.04035	1.05899	1.07957
4	1.27359	1.41879	1.61307	1.92691	4	1.04035	1.05982	1.08099	1.10391
6	1.43938	1.61307	1.84084	2.20463	6	1.05899	1.08099	1.10438	1.12937
8	1.71214	1.92691	2.20463	2.64473	8	1.07957	1.10391	1.12937	1.15628

$\frac{M^{(0)}_{ee'}(100)}{M^{(0)}_{ee'}(cft)}$	2	4	6	8	$\frac{\mathcal{M}^{(0)}_{ee'}(400)}{\mathcal{M}^{(0)}_{ee'}(cft)}$	2	4	6	8
2	1.00272	1.00459	1.00664	1.00885	2	1.00043	1.00071	1.00103	1.00137
4	1.00459	1.00675	1.00907	1.01151	4	1.00071	1.00105	1.0014	1.00178
6	1.00664	1.00907	1.0116	1.01426	6	1.00103	1.0014	1.00179	1.0022
8	1.00885	1.01151	1.01426	1.01709	8	1.00137	1.00178	1.0022	1.00263

$\frac{X^{(0)}_{ee'}(5)}{X^{(0)}_{ee'}(cft)}$	2	4	6	8	$\frac{X^{(0)}_{ee'}(20)}{X^{(0)}_{ee'}(cft)}$	2	4	6	8
2	1.28946	1.42714	1.60523	1.89334	2	1.1113	1.15228	1.19097	1.22861
4	1.42714	1.56675	1.75409	2.06284	4	1.15228	1.18898	1.22491	1.26065
6	1.60523	1.75409	1.9584	2.29906	6	1.19097	1.22491	1.25898	1.29343
8	1.89334	2.06284	2.29906	2.69597	8	1.22861	1.26065	1.29343	1.32697

$\frac{X^{(0)}_{ee'}(100)}{X^{(0)}_{ee'}(cft)}$	2	4	6	8	$\frac{X^{(0)}_{ee'}(400)}{X^{(0)}_{ee'}(cft)}$	2	4	6	8
2	1.03837	1.052	1.06441	1.07597	2	1.01529	1.02071	1.02562	1.03019
4	1.052	1.06393	1.07517	1.08587	4	1.02071	1.02544	1.02988	1.03409
6	1.06441	1.07517	1.08556	1.09557	6	1.02562	1.02988	1.03397	1.0379
8	1.07597	1.08587	1.09557	1.10504	8	1.03019	1.03409	1.0379	1.0416

Table 1: Ratios of MSFT-regulated and CFT Neumann coefficients.

$$V_{00}(cft) \equiv -l_s^2 N_{00}\,, \tag{G.10}$$

$$\begin{aligned} X^{(0,\pm)}_{nm}(cft) &\equiv m\tilde{N}^{(0,\pm)}_{nm}\,, \\ X^{(0,\pm)}_{m0}(cft) &\equiv m\tilde{N}^{(0,\pm)}_{0m}\,. \end{aligned} \tag{G.11}$$

The sign factor in $\mathcal{M}^{(0,\pm)}(cft)$ comes in because we include the multiplication of $C_{nm} = (-1)^n\delta_{n,m}$ in the MSFT definition.

G.2 Ratios of MSFT-regulated and CFT Neumann coefficients

The MSFT-regulated Neumann coefficients are discussed in sections 3.2, 3.3 and A.2.4. The numerical ratios $\mathcal{M}^{(0)}_{ee}(N)\,/\mathcal{M}^{(0)}_{ee'}(cft)$ and $X^{(0)}_{ee}(N)\,/X^{(0)}_{ee'}(cft)$ for $e, e' = 2, 4, 6, 8$ at $N = 5, 20, 100, 400$, shows the convergence of these to the CFT values in the large-N limit. (Table 1)

JHEP07(2003)027

H. Twist and $\mathrm{SU}(1,1)$ in the Siegel gauge

The twist operator is usually given by $\hat{\Omega}=(-1)^{L_0}$. In MSFT this becomes

$$\hat{\beta}_{\Omega}\hat{A}(\bar{x},x_e,p_e;\xi_0,x_o,p_o,y_o,q_o)=\hat{A}(\bar{x},x_e,-p_e;\xi_0,-x_o,p_o,-y_o,q_o)\,. \tag{H.1}$$

This follows from

$$\left\langle x_0,x_e,x_o;c_0,x_e^{gh},x_o^{gh},y_e^{gh},y_o^{gh}\right|\hat{\Omega}=\left\langle x_0,x_e,-x_o;c_0,x_e^{gh},-x_o^{gh},y_e^{gh},-y_o^{gh}\right|. \tag{H.2}$$

When we use the even variables x_e^b,p_e^b,x_e^c,p_e^c, we have the expression

$$\hat{\beta}_{\Omega}\hat{A}(\bar{x},x_e,p_e;\xi_0,x_e^b,p_e^b,x_e^c,p_e^c)=\hat{A}(\bar{x},x_e,-p_e;\xi_0,-x_e^b,p_e^b,-x_e^c,p_e^c)\,. \tag{H.3}$$

The $\mathrm{SU}(1,1)$ generators are given by

$$\begin{aligned}\hat{\mathcal{G}}\,:&=\sum_{n=1}^{\infty}(\hat{c}_{-n}\hat{b}_n-\hat{b}_{-n}\hat{c}_n)\qquad(=N_{\mathrm{gh}}-(\hat{c}_0\hat{b}_0+1))\,,\\ \hat{X}\,:&=-\sum_{n=1}^{\infty}n\hat{c}_{-n}\hat{c}_n\,,\qquad \hat{Y}:=\sum_{n=1}^{\infty}\frac{1}{n}\hat{b}_{-n}\hat{b}_n\end{aligned} \tag{H.4}$$

by oscillator representation [30, 31]. In MSFT we translate them as

$$\begin{aligned}\hat{\beta}_{\hat{\mathcal{G}}}&=\sum_{o>0}\left(y_o\frac{\partial}{\partial y_o}-x_o\frac{\partial}{\partial x_o}+p_o\frac{\partial}{\partial p_o}-q_o\frac{\partial}{\partial q_o}\right)=\sum_{e>0}\left(x_e^c\frac{\partial}{\partial x_e^c}-x_e^b\frac{\partial}{\partial x_e^b}+p_e^b\frac{\partial}{\partial p_e^b}-p_e^c\frac{\partial}{\partial p_e^b}\right),\\ \hat{\beta}_{\hat{X}}&=i\sum_{o>0}\left(y_o\kappa_o\frac{\partial}{\partial x_o}-p_o\kappa_o\frac{\partial}{\partial q_o}\right)=i\sum_{e>0}\left(x_e^c\frac{\partial}{\partial x_e^b}-p_e^b\frac{\partial}{\partial p_e^c}\right),\\ \hat{\beta}_{\hat{Y}}&=i\sum_{o>0}\left(x_o\kappa_o^{-1}\frac{\partial}{\partial y_o}-q_o\kappa_o^{-1}\frac{\partial}{\partial p_o}\right)=i\sum_{e>0}\left(x_e^b\frac{\partial}{\partial x_e^c}-p_e^c\frac{\partial}{\partial p_e^b}\right)\end{aligned} \tag{H.5}$$

on the fields in the Siegel gauge. These operators satisfy the $su(1,1)$ algebra:

$$[\hat{\beta}_{\hat{X}},\hat{\beta}_{\hat{Y}}]=-\hat{\beta}_{\hat{\mathcal{G}}}\,,\qquad[\hat{\beta}_{\hat{\mathcal{G}}},\hat{\beta}_{\hat{X}}]=2\hat{\beta}_{\hat{X}}\,,\qquad[\hat{\beta}_{\hat{\mathcal{G}}},\hat{\beta}_{\hat{Y}}]=-2\hat{\beta}_{\hat{Y}}\,. \tag{H.6}$$

They are derivations with respect to the Moyal $\star$ product:

$$\hat{\beta}_{\hat{\mathcal{O}}}(A_1\star A_2)=(\hat{\beta}_{\hat{\mathcal{O}}}A_1)\star A_2+A_1\star(\hat{\beta}_{\hat{\mathcal{O}}}A_2)\,,\qquad\hat{\mathcal{O}}=\hat{\mathcal{G}},\hat{X},\hat{Y}\,. \tag{H.7}$$

In fact, the above $su(1,1)$ generators (H.5) are inner derivations

$$\begin{aligned}\hat{\beta}_{\hat{\mathcal{O}}}A&=[\beta_{\hat{\mathcal{O}}},A]_{\star}\,,\qquad\hat{\mathcal{O}}=\hat{\mathcal{G}},\hat{X},\hat{Y}\,,\\ \beta_{\hat{\mathcal{G}}}&=\frac{1}{\theta'}\sum_{o>0}(y_oq_o-x_op_o)=\frac{1}{\theta'}\sum_{e>0}\left(x_e^cp_e^c-x_e^bp_e^b\right),\\ \beta_{\hat{X}}&=\frac{i}{\theta'}\sum_{o>0}y_o\kappa_op_o=\frac{i}{\theta'}\sum_{e>0}x_e^cp_e^b,\qquad\beta_{\hat{Y}}=\frac{i}{\theta'}\sum_{o>0}x_o\kappa_o^{-1}q_o=\frac{i}{\theta'}\sum_{e>0}x_e^bp_e^c\,.\end{aligned}$$

By eqs. (H.3), (H.7), we can restrict solutions of the equations of motion (4.15) to the twist even and SU(1,1) singlet sector:

$$\hat{\beta}_{\hat{\Omega}} A(\xi) = A(\xi)\,, \qquad \hat{\beta}_{\hat{\mathcal{O}}} A(\xi) = 0\,, \qquad \hat{\mathcal{O}} = \hat{\mathcal{G}}, \hat{X}, \hat{Y} \tag{H.8}$$

in the Siegel gauge consistently. In fact, we note that in the Siegel gauge

$$\begin{aligned}
[\hat{\beta}_{\hat{\mathcal{O}}}, L_0] &= 0\,, \qquad \hat{\mathcal{O}} = \hat{\Omega}, \hat{\mathcal{G}}, \hat{X}, \hat{Y}\,, \\
\hat{\beta}_{\hat{\Omega}}(A \star A) &= (\hat{\beta}_{\hat{\Omega}} A)\tilde{\star}(\hat{\beta}_{\hat{\Omega}} A) = (\hat{\beta}_{\hat{\Omega}} A) \star (\hat{\beta}_{\hat{\Omega}} A)\,, \qquad \tilde{\star} := \star|_{-\theta,-\theta'}
\end{aligned} \tag{H.9}$$

from eqs. (4.10), (4.11). This condition (H.8) can be used to search for the nonperturbative tachyon vacuum. [30]

In MSFT, it is convenient to note the monoid structure (section 3). It consists of gaussian Moyal fields: $A_{\mathcal{N},M,\lambda} = \mathcal{N} e^{-\bar{\xi} M \xi - \bar{\xi}\lambda}$. We can consider the twist and SU(1,1) symmetric class within the monoid. From eqs.(H.3), (H.5) the restriction by this symmetry (H.8) of a monoid element $A_{\mathcal{N},M,\lambda}(\xi)$ is given by

$$M = \varepsilon M' := \begin{pmatrix} 0 & M' \\ -M' & 0 \end{pmatrix}, \qquad M' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad \bar{A} = A\,, \qquad \bar{B} = B\,, \qquad \lambda = 0 \tag{H.10}$$

in the basis $\bar{\xi} = (x_e^b, p_e^b, x_e^c, p_e^c)$.[27] Namely, the coefficient matrix of the quadratic term in the exponent becomes block diagonal and symmetric. For example, the perturbative vacuum and butterfly states (4.22), (E.2) are of the form of (H.10). Their τ-evolved gaussians are also in this class (4.31). We note that this class of gaussian is not closed within the monoid because of twist operator which changes the sign of noncommutative parameters θ, θ' in the Moyal $\star$ product, while the SU(1,1)-symmetry is conserved in the Siegel gauge by eq. (H.7).

References

[1] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[2] D.J. Gross and A. Jevicki, *Operator formulation of interacting string field theory*, *Nucl. Phys.* **B 283** (1987) 1; *Operator formulation of interacting string field theory. 2*, *Nucl. Phys.* **B 287** (1987) 225.

[3] J. Bordes, H.-M. Chan, L. Nellen and S.T. Tsou, *Half string oscillator approach to string field theory*, *Nucl. Phys.* **B 351** (1991) 441;
A. Abdurrahman, F. Anton and J. Bordes, *Half string oscillator approach to string field theory (ghost sector 2)*, *Nucl. Phys.* **B 411** (1994) 693;
A. Abdurrahman and J. Bordes, *The relationship between the comma theory and Witten's string field theory. i*, *Phys. Rev.* **D 58** (1998) 086003.

[4] L. Rastelli, A. Sen and B. Zwiebach, *Half strings, projectors and multiple D-branes in vacuum string field theory*, *J. High Energy Phys.* **11** (2001) 035 [hep-th/0105058].

[27] If we use the odd basis (x_o, p_o, y_o, q_o), this condition becomes $\bar{A}\kappa_o = \kappa_o A$, $\kappa_o \bar{B} = B\kappa_o$.

[5] D.J. Gross and W. Taylor, *Split string field theory. i*, *J. High Energy Phys.* **08** (2001) 009 [hep-th/0105059]; *Split string field theory. ii*, *J. High Energy Phys.* **08** (2001) 010 [hep-th/0106036].

[6] T. Kawano and K. Okuyama, *Open string fields as matrices*, *J. High Energy Phys.* **06** (2001) 061 [hep-th/0105129].

[7] I. Bars, *Map of Witten's $*$ to Moyal's $*$*, *Phys. Lett.* **B 517** (2001) 436 [hep-th/0106157].

[8] I. Bars and Y. Matsuo, *Associativity anomaly in string field theory*, *Phys. Rev.* **D 65** (2002) 126006 [hep-th/0202030].

[9] I. Bars and Y. Matsuo, *Computing in string field theory using the Moyal star product*, *Phys. Rev.* **D 66** (2002) 066003 [hep-th/0204260].

[10] I. Bars, I. Kishimoto and Y. Matsuo, *String amplitudes from Moyal string field theory*, *Phys. Rev.* **D 67** (2003) 066002 [hep-th/0211131].

[11] I. Bars, *Msft: Moyal star formulation of string field theory*, hep-th/0211238.

[12] I. Bars, I. Kishimoto and Y. Matsuo, *Analytic study of nonperturbative solutions in open string field theory*, *Phys. Rev.* **D 67** (2003) 126007 [hep-th/0302151].

[13] L. Rastelli, A. Sen and B. Zwiebach, *Star algebra spectroscopy*, *J. High Energy Phys.* **03** (2002) 029 [hep-th/0111281].

[14] M.R. Douglas, H. Liu, G. Moore and B. Zwiebach, *Open string star as a continuous Moyal product*, *J. High Energy Phys.* **04** (2002) 022 [hep-th/0202087].

[15] I.Y. Arefeva and A.A. Giryavets, *Open superstring star as a continuous Moyal product*, *J. High Energy Phys.* **12** (2002) 074 [hep-th/0204239];
C.-S. Chu, P.-M. Ho and F.-L. Lin, *Cubic string field theory in PP-wave background and background independent Moyal structure*, *J. High Energy Phys.* **09** (2002) 003 [hep-th/0205218];
D.M. Belov and A. Konechny, *On continuous Moyal product structure in string field theory*, *J. High Energy Phys.* **10** (2002) 049 [hep-th/0207174].

[16] T.G. Erler, *Moyal formulation of Witten's star product in the fermionic ghost sector*, hep-th/0205107.

[17] A.B. Balantekin and I. Bars, *Dimension and character formulas for Lie supergroups*, *J. Math. Phys.* **22** (1981) 1149; *Representations of supergroup*, *J. Math. Phys.* **22** (1981) 1810; *Branching rules for the supergroup* $\mathrm{SU}(N/M)$ *from those of* $\mathrm{SU}(N+M)$, *J. Math. Phys.* **23** (1982) 1239;
I. Bars, *Supergroups and their representations*, *Lectures Appl. Math.***21** (1983) 17.

[18] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization. 1. deformations of symplectic structures*, *Ann. Phys. (NY)* **111** (1978) 61; *Deformation theory and quantization. 2. Physical applications*, *Ann. Phys. (NY)* **111** (1978) 111.

[19] M. Bordemann, *On the deformation quantization of superpoisson brackets*, q-alg/9605038.

[20] H. Omori, Y. Meada, N. Miyazaki and A. Yoshioka, *Singular systems of exponential functions*, in proceedings of the workshop *Noncommutative differential geometry and its applications to physics*, Shonan, Japan, June 1999, *Mathematical Physics Studies* **23** (1999) 169.

[21] I. Kishimoto, *Some properties of string field algebra*, *J. High Energy Phys.* **12** (2001) 007 [hep-th/0110124].

[22] K. Okuyama, *Siegel gauge in vacuum string field theory*, *J. High Energy Phys.* **01** (2002) 043 [hep-th/0111087].

[23] K. Okuyama, *Ghost kinetic operator of vacuum string field theory*, *J. High Energy Phys.* **01** (2002) 027 [hep-th/0201015].

[24] K. Itoh, K. Ogawa and K. Suehiro, *Brs invariance of Witten's type vertex*, *Nucl. Phys.* **B 289** (1987) 127.

[25] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Ghost structure and closed strings in vacuum string field theory*, hep-th/0111129.

[26] M. Schnabl, *Anomalous reparametrizations and butterfly states in string field theory*, *Nucl. Phys.* **B 649** (2003) 101 [hep-th/0202139].

[27] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Star algebra projectors*, *J. High Energy Phys.* **04** (2002) 060 [hep-th/0202151].

[28] T. Okuda, *The equality of solutions in vacuum string field theory*, *Nucl. Phys.* **B 641** (2002) 393 [hep-th/0201149].

[29] W. Taylor, *Perturbative diagrams in string field theory*, hep-th/0207132.

[30] D. Gaiotto and L. Rastelli, *Experimental string field theory*, hep-th/0211012.

[31] B. Zwiebach, *Trimming the tachyon string field with* $\mathrm{SU}(1,1)$, hep-th/0010190.

[32] V.A. Kostelecky and R. Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, *Phys. Rev.* **D 63** (2001) 046007 [hep-th/0008252].